The corresponding simply connected group is the group of unipotent upper triangular 3 by 3 matrices: the exponential map is a bijection because the exponential and logarithm maps are polynomials. The center of the simply connected group is R, and there is an outer automorphism that acts by recaling the center, so there are two possible groups with this Lie algebra, one simple connected, and one with center S^1. The simply connected one can be represented as upper triangular unipotent matrices, and is the group associated with one of the 8 Thurston geometries (the nilmanifolds: take a quotient of the group by a discrete subgroup, such as the subgroup of matrices with integer coefficients). The other group can be represented as the group of transformations of $L^2(R)$ generated by translations and multiplication by e^{ixy}. These satisfy the Weyl commutation relations. The center is multiplication by constants of absolute value 1. This group has no faithful finite dimensional representations: in any finite dimensional representation the center must act trivially. One way to see this is to observe that any element of the center of a characteristic 0 Lie algebra in the derived algebra must act nilpotently in any finite dimensional representation (chop the representation up into generalized eigenspaces, and then look at the trace on any generalized eigenspace. The trace must be zero as the element is in the derived algebra, so the eigenvalue must be zero.) But the only way a nilpotent element can generate a compact group is if it acts trivially. There are several variations and generalizations of these groups. There is a Heisenberg group of dimension $2n+1$ for any positive integer n associated to a symplectic form of dimension $2n$. We can also define Heisenberg groups over finite fields in a similar way.

Exercise 97 Show that over a finite field of prime order p for p odd, every element of the Heisenberg group has order 1 or p, and the exponential map is a bijection. What happens over the field of order 2?

The universal enveloping algebra of the Heisenberg algebra becomes the ring of polynomials in x and d/dx if we take a quotient by identifing the center of the Heisenberg algebra with the real numbers. This gives a representation of the Lie algebra on the ring of polynomials, with the center acting as scalars. The center of this Lie algebra cannot make up its mind whether it is semisimple or nilpotent: in finite dimensional representations it acts nilpotently, but in the infinite dimensional representations we have described its acts semisimply.

- A is not nilpotent and not semisimple. Both eigenvalues must be the same, and we can normalize A so they are both 1. So we can assume A is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. (Bianchi type IV)

Exercise 98 Show that there is a unique connected Lie group with this Lie algebra, and represent it by 3 by 3 upper triangular matrices. Find the conjugacy classes of this group that are in the 2-dimensional derived subalgebra, and sketch a picture of them, paying careful attention to what happens near the origin (the answer may be slightly stranger than you expect).

- A is semisimple, nonzero, with one eigenvector zero. The Lie algebra is the product of the 1-dimensional abelian Lie algebra with the 2-dimensional non-abelian Lie algebra (Bianchi type III). There are two corresponding Lie groups.

- A is semisimple, nonzero, with real non-zero eigenvectors (Bianchi type VI if the eigenvalues are distinct, type V if they are the same). Here we get an uncountable infinite family of distinct Lie algebras, as we can change the smallest eigenvalue to anything we want, but then the second is determined. There is only one connected group for each of these Lie algebras. If the eigenvalues have sum 0 the Lie algebra has an extra symmetry (Bianchi type VI_0) This is the Lie algebra of isometries of 2-dimensional Minkowski space. This also appears as the group of one of the 8 Thurston geometries, giving the solv manifolds. For example one can take a quotient of this group by a cocompact discrete subgroup. Some of these manifolds are the mapping cyliner of an Anosov map of the 2-torus (given by an integral matrix A with distinct real nonzero eigenvectors whose product is 1).

  Exercise 99 Find an example of an Anosov map. Show how to construct a cocompact discrete subgroup of the Bianchi group VI_0 from any Anosov map.

  Exercise 100 Show that the outer automorphism group of this connected Lie group is dihedral of order 8. (Some elements correspond to time reversal, parity reversal, and changing the sign of the metric of Minkowski space.)

  When the eigenvalues are the same the group consists of translations and dilations of the plane.

- A is semisimple, nonzero, with non-real eigenvectors. Bianchi type VII. Again we get an infinite family of Lie algebras. The simply connected group has trivial center except for the following special case (Bianchi type VII_0): this is the one with imaginary eigenvalues, and is the Lie algebra of isometries of the plane. It has an extra symmetry. There is an obvious connected group with this Lie algebra: we can take orientation-preserving isometries of the plane. However this group is not simply connected, as it has homotopy type the circle, so we can also take its universal cover, or the cover of any order $1, 2, 3, \ldots$. We came across this group earlier as a solvable connected Lie group whose exponential map is not surjective.

  Exercise 101 Show that the real Lie algebras of type VI_0 and VII_0 are not isomorphic, but become isomophic when tensored with the complex numbers.

The remaining cases are where G is not solvable, in which case it must be simple as all groups of smaller dimension are solvable. (Similarly the non-solvable finite group of smallest order is necessarily simple.) We will postpone the classification of these as this will be easier when we have developed more theory, and just state the result. There are 2 possible Lie algebras, $su(2)$ (Bianchi type IX) and $sl_2(R)$ (Bianchi type VIII). The first has simply connected group $SU(2)$ with center of order 2, so we get two possible Lie groups (one is $SO_3(R)$). For the other there are two obvious groups $SL_2(R)$ and the quotient by its center $PSL_2(R)$. However there are infinitely many other groups because $SL_2(R)$ is not simply connected: its fundamental group is Z so we can take its universal cover (which also has fundamental group Z) and quotient out by any subgroup of Z. These covers have no faithful finite dimensional representations. The double cover of $SL_2(Z)$ appears in the theory of modular forms of half-integral weight and it called the metaplectic group. It has a representation called he metaplectic representation that we will construct later in the course. The other covers of $PSL_2(R)$ do not seem to appear very often.

The two Lie algebras have the same complexification. This means that the corresponding real Lie algebras or groups are closely related: for example, the finite dimensional complex representation theory of $su(2)$ is essentially the same as that of $sl_2(\mathbb{R})$. However in some ways they are quite different: for example, the irreducible unitary representations of $SU(2)$ are all finite dimensional, while the non-trivial irreducible unitary representations of $SL_2(\mathbb{R})$ are all infinite dimensional.

Exercise 102 Show that the groups $SU(1,1)$, $SL_2(\mathbb{R})$, $Sp_2(\mathbb{R})$ (symplectic group), $SO_{1,2}(\mathbb{R})$ all have the same Lie algebra. Which of the groups are isomorphic?

Exercise 103 The 3-dimensional group of orientation-preserving isometries of 2-dimensional hyperbolic space is one of the groups above. Which one? (One way is to identify hyperbolic space with one of the components of norm 1 vectors in $\mathbb{R}^{1,2}$.)

Exercise 104 Identify the 3-dimensional group of Moebius transformations (invertible conformal transformations of the unit disk in the complex plane) with one of the groups on the list above.

Exercise 105 Show that $\mathbb{R}^3$ with the usual cross-product of vectors is a Lie algebra, and identify it with one of the Bianchi LIe algebras.

Exercise 106 Identify the 3-dimensional Lie algebras of matrices of the forms

$$\begin{pmatrix} 0 & 0 & 0 \\ a & b & c \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} c & 0 & c \\ a & 0 & b \\ 0 & 0 & c \end{pmatrix}$$

with Lie algebras on the list above.

Dimension 4: The classification can be (and has been) pushed beyond dimension 3, but becomes rather tiresome. The problem is that, as suggested by the 3-dimensional case, there are huge numbers of rather uninteresting solvable groups and Lie algebras, which just seem to form a complicated mess. In higher dimensions one just gives up on classifying the sovable ones. We will later prove Levi's theorem that any finite dimensional Lie algebra is a semidirect product of a solvable normal Lie algebra with a product of simple Lie algebras, and will classify the simple ones. So in some sense the finite dimensional Lie algebras can be classified modulo the solvable ones.

Exercise 107 Classify the complex Lie algebras of dimension at most 3. (The Bianchi algebras of types VIII and IX become isomorphic when tensored with the complex numbers.)

Thurston conjectured (and Perelman proved) that 3-manifolds can be cut up in a certain way into 3-manifolds with one of 8 geometries. Five of the eight Thurston geometries in 3 dimensions are the obvious ones: 3-dimensional flat, spherical or hyperbolic space, or the product of 2-dimensional spherical or hyperbolic space with a line. The remaning 3 are those modeled on the 3 groups mentioned above: the nilpotent one, the solvable one related to Minkowski space, and the universal cover of $SL_2(\mathbb{R})$. (Although most of the Thurston geometries can be modeled as left-invariant metrics on 3-dimensional groups this is not true for all of them: there is no 3-dimensional group structure on $S^2 \times \mathbb{R}$.)