8 Picard–Vessiot theory

One of Lie's motivations for studying Lie groups was to extend Galois theory to differential equations, by studying the symmetry groups of differential equations. We will give a very sketchy account of this, missing out most proofs (and for that matter most definitions).

The theorem in Galois theory that a polynomial in characteristic 0 is solvable by radicals if and only if its Galois group is solvable has an analogue for differential equaitons: roughly speaking, a differential equation is solvable by radicals, integration, and exponentiation if and only if its group of symmetries is a solvable algebraic group. This theory was initiated by Picard and Vessiot but it is sometimes hard to tell exactly what they proved as their definitions are somewhat vague. Kolchin gave a rigorous reformulation of their results using the theory of algebraic groups (which he created for this purpose). In particular one needs to distinguish between nilpotent and semisimple abelian groups (which look the same as Lie groups, but are quite different as algebraic groups). The correct analogue of nilpotent Lie algebras is not nilpotent groups but unipotent groups (those such that all eigenvalues of all elements are 1): for example, the group of diagonal matrices is nilpotent but not unipotent.

In this extension of Galois theory, one replaces fields by differential fields: fields with a derivation D. Just as adjoining a root of a polynomial equation to a field gives an extension of fields, adjoining a root of a differential equation to a field gives an extension of differential fields. As in Galois theory, one can form the differential Galois group of an extension $k \subset K$of differential fields as the group of automorphisms of the differential field K fixing all elements of k. Much of the theory of differential Galois groups is quite similar to usual Galois theory: for example, one gets a Galois correspondence between algebraic subgroups of the differential Galois group of an extension and sub differential fields.

Example 90 Suppose we adjoin a root of the equation $df/dx = p(x)$ to the field $k = \mathbb{Q}(x)$ of rational functions over $\mathbb{Q}$. This extension has a group of automorphisms given by the additive group of $\mathbb{Q}$, because we can change f to $f + c$ for some constant of integration c to get an automorphism.

Example 91 Suppose we adjoin a root of the equation $df/dx = p(x)f$ (with solution $\exp(\int p)$) to the field $k = \mathbb{Q}(x)$ of rational functions over $\mathbb{Q}$. This extension has a group of automorphisms given by the multiplicative group of $\mathbb{Q}$, because we can change f to cf for some nonzero constant c to get an automorphism.

The theory applies to homogeneous linear differential equations, so that the set of solutions is a finite-dimensional vector space acted on by the differential Galois group. Equations such as $df/dx = 1/x$ with solutions $\log x$ are not homogeneous so the theory does not apply directly to them, but we can easily turn them into homogeneous equations such as $(d/dx)x(d/df)f = 0$, at the expense of making the space of solutions 2-dimensional rather than 1-dimensional.

Exercise 92 Find the Lie group of automorphisms of the solutions of

$$(d/dx)x(d/df)f = 0$$

and describe its action on the space of solutions.

We will sketch the proof of one of the results of Picard-Vessiot theory, which says roughly that a linear homogeneous differential equation can be solved by radicals, exponentials, and integration if and only if its differential Galois group is solvable.

In one direction this follows by calculating the differential Galois group: each time we take radicals we get a finite cyclic group, each time we take an integral we get a differential Galois group isomorphic to the additive group, and each time we take an exponential we get a differential Galois group isomorphic to the multipicative group. So by repeating such extensions we get a group built out of additive groups, which is solvable.

Conversely, suppose the differential Galois group is solvable. The quotient by the connected component is a finite solvable group, which corresponds to repeatedly taking radicals just as in ordinary Galois theory, so we can assume that the differential Galois group is connected and solvable. Now we apply Lie's theorem on solvable Lie algebras (or more precisely Kolchin's version of it for solvable algebraic groups) so see that the differential Galois group has an eigenvector f in the space of solutions of the differential equation. Also Df has the same eigenvalue as D commutes with the differential Galois group, so Df/f is fixed by the differential Gaois group and so is in the base field. So f satisfies the differential equation $Df = af$ for some a, which can be solved by exponentials and integration.

Example 93 A typical application of differential Galois theory is that Bessel's equation $x^2d^2y/dx^2+xdy/dx+(x^2-\nu^2)y = 0$ cannot be solved using integration and elementary functions unless $\nu - 1/2$ is integral. Except for these special values, the differential Galois group is SL_2 which is not solvable. Finding the differential Galois group is rather too much of a digression, but we can at least get non-trivial upper and lower bounds for it as follows. First, we can show that it lies in SL_2 (rather than just GL_2) by using the Wronskian of the equation, given by $W = \begin{pmatrix} f & g \\ f' & g' \end{pmatrix}$ for two independent solutions f and g. The Wronskian get multiplied by the determinant of a matrix of the differential Galois group, so the elements of the differential Galois group have determinant 1 if and only if the Wronskian is in the base field.

Exercise 94 Show that the Wronskian of $d^2y/dx^2 + p(x)dy/dx + q(x)y = 0$ satisfies the differential equation $dW/dx + p(x)W = 0$. Use this to find the Wronskian of Bessel's equation, and deduce that the differential Galois group lies in the special linear group.

To find a lower bound for the differential Galois group, we observe that monodromy gives elements of this group. (Monodromy means go around a branch point.) Bessel's equation has a branch point at 0, and the two solutions $J_\nu = x^\nu \times$(something holomorphic) and $J_{-\nu} = x^{-\nu} \times \cdots$ for ν not an integer are mutliplied by $e^{\pm 2\pi i\nu}$ by the monodromy, so the differential Galois group contains the diagonal matrix with these entries. When ν is an integer the monodromy is unipotent instead of semisimple: in this case the solutions are J_ν with trivial monodromy, and $Y_\nu =$ (something holomorphic)$+J_\nu\times$(something holomorphic)$\times$ log that has a logarithmic singularity and is changed by a multiple of J_ν by monodromy. So in this case the differential Galois group has a unipotent element of the form $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ generated by monodromy.

For proofs of most of the results discussed here, see Kolchin's papers on differential Galois theory.

9 Lie groups of dimension at most 3

We will find all (real, connected) Lie groups and Lie algebras of dimension at most 3.

Dimension 0: This is the hardest case, as it involves classifying all discrete groups, which is hopeless. Even if we restrict to compact simple groups, the 0-dimensional case is the classification of finite simple groups, which is about a thousand times longer than the classification of compact simple Lie groups of positive dimension. In general any Lie group has a normal closed subgroup consisting of the connected component of the identity, and the quotient is a discrete group. So we just give up on the discrete part, and from now on try to find the connected Lie groups of small dimension.

Dimension 1: The only 1-dimensional Lie algebra is the abelian one. The corresponding simply connected group is just the reals under addition. Other groups come by quotienting by a discrete subgroup of the center: up to equivalence, the only way to do this is to take R/Z. So there are just two 1-dimensional connected Lie groups: The reals and the circle group.

Dimension 2: First we find the Lie algebras. One possibility is that the algebra is abelian. Otherwise the derived algebra has dimension 1 (spanned by $[a, b]$ for any two independent vectors), so we take one element a of a basis to span the derived algebra. For any other vector we have $[a, b]$ is a multiple of a, so by multiplying b by a constant we can assume that $[a, b] = a$. So there is just one non-abelian Lie algebra.

The abelian groups correspond to quotients of R^2 by discrete subgroups (or lattices) in R^2. There are 3 possibilities: the lattice can have rank 0, 1, or 2, giving 3 groups R^2, $R^1 \times S^1$, and $S^1 \times S^1$ (the torus).

Exercise 95 Find the automorphism groups of the 3 connected 2-dimensional abelian Lie groups.

The non-abelian simply connected group is the $ax + b$ group that can be represented as the orientation preserving affine transformations of the real line of the form $x \mapsto ax + b$ for a positive. It also appears acting on the upper half plane by the same formula, and as 2 by 2 matrices of the form $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$ with $a > 0$. The center is trivial, so this is the only non-abelian 2-dimensional Lie group. It is solvable but not nilpotent.

This group has analogues over finite fields that are semidirect products of the additive group of order q by the multiplicative group of order $q - 1$. More generally, we can form the semidirect product of the additive group of order q by any subgroup of the multiplicative group, whcih can be a cyclic group of any order dividing $q - 1$. These groups account for many of the small non-abelian finite groups.

Exercise 96 Show that any nonabelian group of order pq for $p < q$ primes is of this form: more precisely, there are no such groups unless p divides $q - 1$, in which case there is a unique such group, given by a subgroup of the $ax + b$ group.

Dimension 3: This is where things start to get hairy. We find the connected groups by first finding the Lie algebras, and then finding the corresponding simply connected Lie group, and then finding the discrete subgroups of its center. The algebras were classified by Bianchi.

We first assume that G is solvable. We start by showing that G has a normal abelian subalgebra of dimension 2. It certainly has some normal subalgebra of dimension 2 (codimension 1) as G is not perfect. If this is not abelian then it must be the unique non-abelian Lie algebra of dimension 2, so G is a semidirect product of this by a 1-dimensional algebra acting on it. However this 2-dimensional non-abelian Lie algebra has no outer derivations, so the Lie algebra is just a product of the 2-dimensional non-abelian Lie algebra with a 1-dimensional Lie algebra, in which case it has a normal 2-dimensional abelian Lie algebra.

So we see that G can be given as follows: it has a normal 2-dimensional abelian subalgebra, and is a semidirect product by a 1-dimensional algebra acting on it by some transformation A. So G is determined by the 2 by 2 real matrix A. Changing A by conjugation or multiplying it by a non-zero constant does not change the isomorphism class of the Lie algebra. So to classify the solvable Lie algebra of dimension 3, we just have to run through all possible types of 2 by 2 matrices as follows.

- A is zero. The Lie algebra is abelian. (Bianchi type I). There are now 4 possibilities, all products of copies of the circle and the real line.

- A is nilpotent but not zero. The Lie algebra is the Heisenberg algebra (Bianchi type II).