7 Solvable Lie groups

Recall that a solvable group is one all of whose composition factors are abelian. The term comes from Galois theory, where a polynomial is solvable by radicals (and Artin–Schrier extensions in positive characteristic) if and only if its Galois group is solvable. For Lie groups the term solvable has the same meaning, and for Lie algebras it means the obvious variation: the Lie algebra is solvable if all composition factors are abelian Lie algebras.

The main goal of this section is to prove Lie's theorem that a complex solvable Lie algebra of matrices is conjugate to an algebra of upper triangular matrices.

Lie's theorem fails in positive charactersitic, so in proving it we need to make use of some property of matrices that holds in characteristic 0 but not in positive characteristic. One such property is that if the trace of λI vanishes then so does λ; this is used in the following lemma.

Lemma 79 *Suppose that the Lie algebra G over a field of characteristic 0 has an ideal H and acts on the finite dimensional vector space V. Then G acts on each eigenspace of H.*

Proof Recall that an eigenvalue of H is given by some linear form λ on H, and the corresponding eigenspace consists of vectors v such that $h(v) = \lambda(h)v$ for all $h \in H$. Pick some eigenvector of H with eigenvalue λ, and pick some $g \in G$. We need to show that $g(v)$ also has eigenvalue λ. Look at the space W spanned by g, gv, g^2v,... which has an increasing filtration $0 = W_0 \subset W_1 \subset \cdots \subset W_n = W$ where W_i is spanned by W_{i-1} and g^iv. Then each W_i/W_{i-1} is at most 1-dimensional and is acted on by H with eigenvalue λ, because $[g, h]$ is in H. So on W, any element h of H has trace $n\lambda(h)$. In particular $[g, h]$ has trace $n\lambda([g, h])$, so $\lambda([g, h]) = 0$ because $[g, h]$ has trace 0 and n is invertible (this is where we use the characteristic 0 assumption). But $\lambda([g, h]) = 0$ implies that $hgv = ghv = g\lambda(h)v$, so gv is an eigenvalue of H with eigenvalue λ, which is what we were trying to prove. □

This lemma really does fail in infinite dimensions or in characteristic $p > 0$. For example, we can take the nilpotent Lie algebra spanned by the operators 1, x, d/dx which acts on $k[x]$. Then 1 is an eigenvalue of d/dx, but $x1$ is not. In characteristic p we can take the finite dimensional quotient $k[x]/(x^p)$. It is clear from the proof that it holds in characteristic $p > 0$ provided the vector space V has dimension less than p. this is quite a common phenomenon: results true in characteristic 0 are often true in characteristic $p > 0$ provided we stick to vector spaces of dimension less than p.

Theorem 80 *Lie's theorem. If a solvable Lie algebra G over an algebraically closed field of characteristic 0 acts on a non-zero finite-dimensional vector space, it has an eigenvector.*

Proof If G is nonzero, then as it is solvable we can find an ideal H of codimension 1. By induction on the dimension of G there is an eigenspace W of H for some eigenvalue of H. If g is any element of G not in H then by the previous lemma g acts on W, and as we are working over an algebraically closed field we can find some eigenvector of g on W. This is an eigenvector of G because G is spanned by g and H. □

By repeatedly applying this theorem, we see that the Lie algebra fixes a flag. So solvable Lie subalgebras of $M_n(C)$ are conjugate to subalgebras of the Lie algebra of upper triangular matrices.

Another way of stating Lie's theorem is that any irreducible representation of a finite-dimensional complex solvable Lie algebra is 1-dimensional. This does not mean that their representation theory is trivial. Non-abelian solvable complex Lie algebras have plenty of infinite dimensional irreducible representations. And even finite dimensional representations are hard to study, because there are plently of indecomposbale representations that are not irreducible. In fact, even for abelian Lie algebras of dimension 2, the finite dimensional indecomposable representations are very hard to classify. We will see later that the representation theory of simple Lie algebras is much easier, because we do not ahve this problem: all indecomposable representations are irreducible.

If we examine the proof, we see that Lie's theorem still holds in positive characteristic provided the dimension of the vector space is less than the characteristic.

Example 81 The solvable (in fact nilpotent) Lie algebra spanned by the operators 1, x, d/dx acts on $k[x]$, and has no eigenvectors. In characteristic p is acts on the finite dimensional quotient $k[x]/(x^p)$, but has no eigenvectors; in fact the action is irreducible. So Lie's theorem fails in characteristic $p > 0$ and in infinite dimensions.

Corollary 82 *The derived subalgebra of a finite dimensional solvable Lie algebra over a field of characteristic 0 is nilpotent.*

Proof We can extend the field to be algebraically closed. In this case the corollary follows from Lie's theorem, because the Lie algebra can be assume to be upper triangular, in which case its derived algebra consists of strictly upper triangular matrices and is therefore nilpotent. □

Although Lie's theorem fails in positive characteristic for Lie algebras, it still holds for solvable algebraic groups in any characteristic: this is Kolchin's theorem. (However it fails for solvable connected Lie groups: these are not necessarily isomorphic to groups of (upper triangular) matrices.) More generally still, Borel proved that any solvable algebraic group acting on a projective variety (over an algebraically closed field) has a fixed point. The special case when the projective variety is projective space is Kolchin's theorem.

Example 83 There are obvious analogues of Lie's theorems for connected solvable Lie groups of matrices. However for disconnected solvable groups the conclusions do not hold. For example, the symmetric group S_3 acting on its irreducible 2-dimensional representation has no eigenvectors. And the derived subgroup of a solvable finite group is usually not nilpotent: an example is the solvable symmetric group S_4 whose derived subgroup is the alternating group A_4.

Example 84 Lie's theorem shows that in some sense solvable connected Lie groups are not too far from nilpotent ones: they are given by sticking an abelian group on top of a nilpotent one. For (disconnected) finite groups, the solvable ones can be much more complicated than nilpotent ones. For example, a typical example of a smallish solvable group is $GL_2(\mathbb{F}_3)$ of order 48 with the chain of normal subgroups $1 \supset \mathbb{Z}/2\mathbb{Z} \subset Q_8 \subset SL_2(\mathbb{F}_3) \subset GL_2(F_3)$ with quotients of orders 2, 4, 3, 2. Larger finite solvable groups tend to be a similar but more complicated mess, and are rather hard to work with. The relatively easy structure of solvable connected groups is one of the reasons that connected Lie groups are easier to handle than finite groups.

Exercise 85 Which well-known group is $PGL_2(\mathbb{F}_3)$ isomorphic to?

Exercise 86 Over a field k of characteristic $p > 0$, show that the semidirect product of the 3-dimensional Lie algebra $\{1, x, d/dx\}$ by the p-dimensional abelian Lie algebra $k[x]/(x^p)$ is solvable, but its derived algebra is not nilpotent. This shows that the corollary above fails in positive characteristic.

Although in some ways solvable Lie algebras are not too far from niloptent ones, their behavior can be much more complicated. For example, for any connected nilpotent Lie algebra, the exponential map to the simply connected group is an isomorphism (of sets). For example, we can use the BCH formula to define a Lie group structure on the Lie algebra.

Exercise 87 Show that if a a Lie group G has a connected central subgroup H, and the exponential map is surjective for G/H, then it is surjective for G. Deduce that the exponential map is surjective for connected nilpotent Lie groups.

It is very plausible that a similar result holds for solvable Lie algebras. For example, if a Lie algebra $\mathfrak{g}$ has a normal subalgebra $\mathfrak{h}$ such that the exponential maps take $\mathfrak{g}$ and $\mathfrak{h}$ onto their simply connected Lie groups then it seems almost obvious that the same is true for $\mathfrak{g}$, which would prove it for all solvable Lie algebras. Rather surprisingly, this is in fact sometimes false: the exponential map for a solvable Lie algebra need not map onto the simply connected group.

Example 88 Let $\mathfrak{g}$ be the Lie algebra of (orientation preserving) isometries of the Euclidian plane. If we identify the Euclidian plane with the complex numbers and rotations with multiplication by complex numbers of absolute value 1, then the group G can be thought of as the matrices $\begin{pmatrix} e^{it} & z \\ 0 & 1 \end{pmatrix}$ for t real and z complex. The Lie algebra consists of matrices of the form $\begin{pmatrix} it & z \\ 0 & 0 \end{pmatrix}$ and the exponential map takes this to $\begin{pmatrix} e^{it} & z(e^{it}-1)/(it) \\ 0 & 1 \end{pmatrix}$. (Recall the fast way to see this: exp of an $n \times n$ matrix is a polynomial of degree less than n in it.) Examining this we see that the exponential map is surjective, but not injective. This is easy to fix: we can make it injective by replacing the Lie group by its universal cover (the fundamental group is just $\mathbb{Z}$). So what is the problem? The problem is that the exponential map is surjective the group G, but is NOT surjective for the universal cover of G. To see this, notice that points of G of the form $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ are in the image of only 1 point under the exponential map. So only one of their inverse images in the universal cover can be in the image of the exponential map. So there is no group such that the exponential map of $\mathfrak{g}$ is an isomorphism: it fails to be either injective or surjective (or both, if we take a non-trivial finite cover).

This problem is closely related to the fact that the Lie algebra has elements with non-zero purely imaginary eigenvalues in the adjoint representation.

Exercise 89 Show that the universal cover of the group of orientation-preserving isometries of the plane can be represented faithfully as a group of 3 by 3 matrices.

$$\begin{pmatrix} e^{it} & 0 & z \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

Use Kolchin's theorem that a solvable algebraic group can be diagonalized (over $\mathbb{C}$) to show that it cannot be represented faithfully as an algebraic group.

There are several other ways in which solvable Lie groups are fundamentally more complicated than nilpotent ones. We will see later that left-invariant Haar measures on nilpotent Lie groups are right-invariant, but this need not be true for solvable Lie groups. Also the representation theory of solvable Lie algebras can be a lot wilder than the representation theory of nilpotent ones, in the sense that von Neumann algebras not of type I can appear. A related fact is that the coadjoint orbit space (the space of orbits of the group on the dual of the Lie algebra) of a solvable Lie algebra can have unpleasant topological properties: it need not be T_0 for example.