A useful way of thinking of free groups is that they are the fundamental groups of (connected) graphs with base points. Given such a graph we can obtain an independent set of generators for its free fundamental group by picking a maximal tree. The remaining edges then correspond to generators of the fundamental group as follows: given such an edge, start at the basepoint, travel along the tree to one end of the edge, go along the edge, then go back along the tree to the basepoint.

Lemma 66 *Any subgroup of a free group is free. More precisely, an index m subgroup of a free group on n generators is free on $m(n-1)+1$ generators.*

Proof Represent the free group as the fundamental group of a graph with n loops. Then a subgroup of index m is the fundamental group of the corresponding m-fold connected cover. Since this is also a tree, its fundamental group is also free.

To count the number of generators, observe that the number of generators of the fundamental group of a graph is $1-\chi$ where χ is the Euler characteristic (number of vertices minus number of edges). Since the Euler characteristic gets multiplied by m when we take an m-fold cover, this gives the number of generators of a subgroup. □

Exercise 67 Consider the action of the free group on 2 generators a, b on 3 points 1, 2, 3 such that a and b act as the transposition (12) and (13). Find a set of four generators for the free subgroup fixing 1. (Draw the graph with three vertices 1, 2, 3 and four edges giving the actions of a and b on the vertices, then pick a maximal tree (with 2 edges) then find the four generators by starting at 1, running along the tree, across and edge, and back along the tree.)

Exercise 68 How many subgroups of index 3 does the free group on 2 generators have? (The subgroups correspond to transitive actions on 3 points, one of which is marked.) How many triple covers does a figure 8 have?

Now we show that free Lie algebras and free Lie groups are closely related. This may be a little surprising, because these correspond to connected and discrete groups, which in some sense are opposite to each other. Given a free group F, we can form its descending central series $F_0 \supseteq F_1 \supseteq \cdots$, with $F_{i+1} = [F_i, F]$, the group generated by commutators.

If a group has a descending central series $G_0 \supset G_1 \cdots$ we can construct a graded Lie ring from it as follows. The Lie ring will be $G_0/G_1 \oplus G_1/G_2 \oplus \cdots$. The additive structure of the ring is just given by the (abelian) group structure on each quotient. The Lie bracket is given by the commutator $[a, b] = a^{-1}b^{-1}ab$. The key point is to check that the Jacobi identity holds. This follows from Philip Hall's identity:

Exercise 69

$$[[x, y^{-1}], z]^y \cdot [[y, z^{-1}], x]^z \cdot [[z, x^{-1}], y]^x = 1$$

Exercise 70 Check that $G_0/G_1 \oplus G_1/G_2 \oplus \cdots$ is a Lie ring.

Theorem 71 *The Lie ring of the descending central series of the free Lie group on n generators is the free Lie ring on these generators.*

Proof First, there is an obvious homomorphism from the free Lie ring to the Lie ring of the free group, by the universal property of the free ring. To prove this is an isomorphism we want to construct a map in the other direction. We do this as follows.

We map each generator A of the free group to $\exp(a)$ in the rational completed universal enveloping algebra of the free Lie ring, where a is the generator of the free Lie ring corresponding to the generator A of the free group. This extends to a homomophism f of groups by the universal property of a free group. If A is in F_n then $f(A)$ is of the form $1 + a_{n+1} + a_{n+2} + \cdots$ where a_i has degree i in the universal enveloping algebra. We define the image of A to be the element a_{n+1}. We can check that this is primitive (as the log of a group-like element is primitive) and integral, so an element of the free Lie algebra. We can also check that this preserves addition and the Lie bracket and so gives a Lie algebra homomorphism from the Lie ring of the free group. This gives the desired inverse map, so proves that the Lie ring of the free group is the free Lie algebra. $\square$

Exercise 72 Show that free groups are residually nilpotent. Show that free Lie algebras are residually nilpotent.

So the relation between the free group and the free Lie algebra on some generators is given as follows. The Lie ring of the free group is the free Lie ring on the generators. The group generated by the elements $\exp(a)$, as a runs through generators for a free Lie ring, is the free group.

6 Nilpotent Lie groups

The main result about nilpotent Lie algebras is Engel's theorem, due to Friedrich Engel (not to be confused with the philosopher Friedrich Engels).

Theorem 73 *(Engel) Suppose that g is a Lie algebra of nilpotent endomorphisms of a non-zero finite dimensional vector space V. Then V has a nonzero vector fixed by g.*

Proof We use induction on the dimension of g. The main step is to show that g has an ideal h of codimension 1 (unless g is 0). So fix any proper nonzero subalgebra h of g. Then h acts on g by nilpotent endomorphisms, and so acts on the vector space g/h by nilpotent endomorphisms. By induction there is a nonzero element of g/h killed by h, so if h has codimension greater than 1 we can add this to h and repeat until h has codimension 1. In this case h is an ideal of g.

Now look at the subspace W of V fixed by all elements of h, which is nonzero by induction. This is acted on by the 1-dimensional Lie algebra g/h as h is an ideal, and as g/h acts by a nilpotent endomorphism of W there must be a non-trivial fixed vector. $\square$

This theorem shows that if g is a Lie algebra of nilpotent endomorphisms of V, then there is a flag $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ such that g acts trivially on each V_i/V_{i-1}. (Take V_1 to be the vectors fixed by g and apply induction to V/V_1). In other words V has a basis so that g is strictly upper triangular. Conversely any strictly upper triangular Lie algebra consists of nilpotent endomorphisms.

We would like to say that a Lie algebra is nilpotent if all elements are represented by nilpotent matrices, but there is a slight problem that this depends on the choice of representation: an 1-dimensional abelian Lie algebra can be represented by either a nilpotent or a non-nilpotent matrix. So instead we use the following definition:

Definition 74 *A Lie algebra g is called nilpotent if it has a central series* $0 = g_0 \subset g_1 \subset \cdots \subset g_n = g$*. This means that each g_i is an ideal, and g fixes all elements of g_i/g_{i-1} (or equivalently that g_i/g_{i-1} is in the center of g/g_{i-1}).*

There are two obvious ways to construct a central series of a group or Lie algebra: we can start at the bottom, and repeatedly quotient out by the center (g_i/g_{i-1} = center of g/g_{i-1}), or we can start at the top and repeatedly take commutators ($g_i = [g, g_{i+1}]$). The first method produces the "largest" central series and the second produces the "smallest". It is also possible to continue the upper and lower central series "transfinitely" but then they are no longer closely related: for example, for the free group or Lie algebra the descending central series becomes trivial after ω steps, but the ascending one never takes off as the center is trivial.

Exercise 75 Find an example of a nilpotent Lie algebra whose ascending and descending central series are not the same. (The smallest example is 4-dimensional.)

A reasonably typical example of a nilpotent Lie algebra is the Lie algebra of all strictly upper triangular matrices. A Lie algebra is nilpotent if and only if it is isomorphic to a Lie algebra of strictly upper triangular matrices. This follows immediately from Engel's theorem if we can show that it has a finite-dimensional faithful representation in which all elements act nilpotently. We will prove this later as a special case of Ado's theorem.

Similarly we define a group to be nilpotent if it has a central series. There is nothing obviously nilpotent in a nilpotent group: the terminology comes from Lie algebras.

Theorem 76 *For finite groups, the nilpotent ones are just the products of groups of prime power order.*

Proof First we show that any group of prime power order is nilpotent. The key step is to show that it has a nontrivial center (if it is nontrivial). For this, look at the partition into conjugacy classes. Each conjugacy class has order (order of G)/(order of subgroup fixing an element), so has order divisible by p if it is not in the center. If G is nontrivial it also has order divisible by p, so the number of elements in the center is divisible by p. So by repeatedly killing the center we see that G has a central series and is nilpotent.

It is trivial to see that a product of two nilpotent groups is nilpotent, so any finite product of groups of prime power order is nilpotent.

Conversely we want to show that if G is nilpotent then it is a product of its Sylow subgroups, or in other words all its Sylow subgroups are normal. If G

is nontrivial then we can find an element of some prime order p in the center generating a subgroup H as G is nilpotent, and by induction G/H is a product of its Sylow subgroups. But if Q is a Sylow q-subgroup of G, then its image in G/H is the unique Sylow q-subgroup of G/H, whose inverse image in G is QH. But since H is in the center of QH there is only one Sylow q-subgroup of QH (either Q or QH depending on whether or not $p = q$) so it must be normalized by G. So all Sylow subgroups of G are normal, so G is a product of its Sylow subgroups. $\square$

For any finite nilpotent group we can construct a finite Lie ring of the same order, as in the previous section. This does not seem to help all that much, as finite Lie rings seem just as messy as finite nilpotent groups.

The naive analogue of Engel's theorem fails for nilpotent groups: for example, the dihedral group of order 8 is nilpotent but has no fixed vectors in its 2-dimensional real representation. However there is an analogue that works: if a p-group acts on a non-zero finite dimensional vector space over a field with p elements then it fixes some vector The proof is similar to the proof that a non-trivial p group has a nontrivial center. There is also a similar analogue for algebraic groups: an algebraic group acting on a nonzero vector space whose elements are unipotent (all eigenvalues 1, or equivalently 1 plus nilpotent) fixes some vector, so is conjugate to a group of upper-triangular matrices with 1's on the diagonal.

There are huge numbers of p-groups of order p^n if n is reasonably large; in fact the number of groups of order less than some number is dominated by groups of order 2^n. In fact we can do this with just 2-step nilpotent groups. Fix two vector spaces V, W over a field (such as the field with p elements). If we are given a skew symmetric bilinear map $[,]$ from $V \times V$ to W then we can make $V \oplus W$ into a nilpotent group by letting the commutator of w_1, w_2 be the element $[w_1, w_2]$ of V. So the number of groups we get is about $p^{\dim(V)^2 \dim(W)/2}$. For groups of order p^n we have $\dim(V) + \dim(W) = n$, so the number of groups is maximized for $\dim(V) = n/3$, $\dim(W) = 2n/3$, and the number of groups is about $p^{2n^3/27}$. Of course we should divide out by groups that are isomorphic, but the number of choices we make is only $p^{\text{something quadratic in } n}$ so this is dominated by the cubic exponent of p and does not reduce the number of groups all that much. The number of groups of order 2^n is 1, 1, 2, 5, 14, 51, 267, 2328, 56092, 10494213, 49487365422, ..., and almost all groups of order less than some large integer are 2-group.

Exercise 77 Classify the groups of order 8. (The 3 abelian ones are obvious; the other two are the dihedral group and the quaternion group. One way to find the non-abelian ones is to start by observing that the center of a nonabelian group has order 2 and the quotient by the center is a Kelin 4-group.)

Trying to classify nilpotent Lie algebras or nilpotent Lie groups of given dimension is just as bad: beyond dimension about 6 or so everything just gets horribly messy.

We saw earlier that in some sense Lie groups are commutative to first order. This suggests that maybe discrete subgroups generated by elements close to the identity will be commutative. This is not quite correct: for example, the group of unipotent upper trianular matrices has non-abelian subgroups generated by elements close to the identity. However Zassenhaus showed that it is essentially correct, except that "abelian" has to be replaced by "nilpotent", which is in some sense very close to "abelian".

Theorem 78 *(Zassenhaus) The identity of a Lie group has a neighborhood U with the following property: any discrete subgroup generated by elements of U is nilpotent.*

Proof The idea of the proof is that elements near the identity almost commute with each other. The commutator of two elements is second order. So if U is small enough then then sets $u_1 = U$, $U_2 = [U, U]$, $U_3 = [[U, U], U]$, will tend to 0 in the sense that they will eventually be in any given neighborhood. If a subgroup H of G is discrete this means that its intersection with U_n for n large is the identity element. If in addition H is generated by elements of U, this means that $[...[g_1, g_2], g_3, ...], g_n] = 1$ for any n elements in the generating set, which implies that H is nilpotent of step n. $\square$

For nilpotent Lie algebras $\mathfrak{g}$ over fields of characteristic 0, the Campbell-Baker-Hausdorff formula converges as it only has a finite number of nonzero terms, so can be used to give $\mathfrak{g}$ a group structure. In particular, if G is a nilpotent Lie group then its universal covering space is a vector space and in particular is contractible. This fails completely for general Lie groups: for example the group S^3 is simply connected so has universal covering space a sphere.

We should at least mention Gromov's theorem that a finitely-generated group has polynomial growth if and only if it has a nilpotent subgroup of finite index.