1 Examples

A typical example of a Lie group is the group $GL_2(\mathbb{R})$ of invertible 2 by 2 matrices, and a Lie group is defined to be something that resembles this. Its key properties are that it is a smooth manifold and a group and these structures are compatible. So we define a Lie group to be a smooth manifold that is also a group, such that the product and inverse are smooth maps. All manifolds will be smooth and metrizable unless otherwise stated.

We start by trying to list all Lie groups.

Example 1 Any discrete group is a 0-dimensional Lie group.

This already shows that listing all Lie groups is hopeless, as there are too many discrete groups. However we can split a Lie group into two: the component of the identity is a connected normal subgroup, and the quotient is discrete. Although a complete description of the discrete part is hopeless, we can go quite far towards classifying the connected Lie groups.

Example 2 The real numbers under addition are a 1-dimensional commutative Lie group. Similarly so is any finite dimensional real vector space under addition.

Example 3 The circle group S^1 of all complex numbers of absolute value 1 is a Lie group, also abelian.

We have essentially found all the connected abelian Lie groups: they are products of copies of the circle and the real numbers. Foe example, the non-zero complex numbers form a Lie group, which (via the exponential map and polar decomposition) is isomorphic to the product of a circle and the reals.

Example 4 The general linear group $GL_n(\mathbb{R})$ is the archetypal example of a non-commutative Lie group. This has 2 components as the determinant can be positive or negative. Similarly we can take the complex general linear group.

The classical groups are roughly the subgroups of general linear groups that preserve bilinear or hermitian forms. The compact orthogonal groups $O_n(\mathbb{R})$ preserve a positive definite symmetric bilinear form on a real vector space. We do not have to restrict to positiver definite forms: in special relativity we get the Lorentz group $O_{1,3}(\mathbb{R})$ preserving an indefinite form. The symplectic group $Sp_{2n}(\mathbb{R})$ preserves a symplectic form and is not compact. The unitary group U_n preserves a hermitian form on C^n and is compact as it is a closed subgroup of the orthogonal group on $\mathbb{R}^{2n}$. Again we do not have to restict to positive definite Hermitean forms, and there are non-compact groups $U_{m,n}$ preserving $|z_1|^2 + \cdots + |z_m|^2 - z_{m+1}^2 - \cdots$.

There are many variations of these groups obtained by tweaking abelian groups at the top and bottom. We can kill off the abelian group at the top of many of them by taking matrices of determinant 1: this gives special linear, special orthogonal r groups and so on. ("Special" usually means determinat 1). Alternatively we can make the abelian group at the top bigger: the general symplectic group GSp is the group of matrices that multiply a symplectic form by a non-zero constant. We can also kill off the abelian group at the bottom (often the center) by quotienting out by it: this gives projective general linear groups and so on. (The word "projective" usually means quotient out by the center, and comes from the fact that the projective general linear group acts on projective space.) Finally we can make the center bigger by taking a central extension. For example, the spin groups are double covers of the special orthogonal groups. The spin group double cover of $SO_3(\mathbb{R})$ can be constructed using quaternions.

Exercise 5 If $z = a + bi + cj + dk$ is a quaternion show that $z\overline{z}$ is real, where $\overline{z} = a - bi - cj - dk$. Show that $z \mapsto |z| = \sqrt{z\overline{z}}$ is a homomorphism of groups from non-zero quaternions to positive reals. Show that the quaternions form a division ring; in other words check that every non-zero quaternion has an inverse.

Exercise 6 Identify $\mathbb{R}^3$ with the set of imaginary quaternions $bj + cj + dk$. Show that the group of unit quaternions S^3 acts on this by conjugation, and gives a homomorphism $S^3 \mapsto SO_3(\mathbb{R})$ whose kernel has order 2.

A typical example of a solvable Lie group is the group of upper triangular matrices with nonzero determinant. (Recall that solvable means the group can be split into abelian groups.) It has a subgroup consisting of matrices with 1s on the diagonal: this is a typical example of a nilpotent Lie group. (Nilpotent means that if you keep killing the center you eventually kill the whole group. We will see later that a connected Lie group is nilpotent if all elements of its lie algebra are nilptent matrices: this is where the name "nilpotent" comes from.)

Exercise 7 Check that these groups are indeed solvable and nilpotent.

Exercise 8 Show that any finite group of prime-power order p^n is nilpotent, and find a non-abelian example of order p^3 for any prime p. (Hint: show that any conjugacy class not in the center has order divisible by p, and deduce that the center has order divisible by p unless the group is trivial.)

Exercise 9 The Moebius group consists of all ismorphisms from the complex unit disk to itself: $z \mapsto (az + b)/(cz + d)$ with $ad - bc = 1$, $a = \overline{d}$, $b = \overline{c}$. Show that this is the group $PSU_{1,1}$. Similarly show that the group of conformal transformations of the upper half plane is $PSL_2(\mathbb{R})$. Since the upper half plane is isomorphic to the unit disc, we see that the groups $PSU_{1,1}$ and $PSL_2(\mathbb{R})$ are isomorphic. This illustrates one of the confusing things about Lie groups: there are a bewildering number of unexpected isomorphisms between them in small dimensions.

Exercise 10 Show that there is a (nontrivial!) homomorphism from $SL_2(\mathbb{R})$ to the group $O_{2,1}(\mathbb{R})$, and find the image and kernel. (Consider the action of the groups $SL_2(\mathbb{R})$ on the 3-dimensional symmetric square $S^2(\mathbb{R}^2)$ and show that this action preserves a quadratic form of signature $(2, 1)$.)

Klein claimed at one point that geometry should be identified with group theory: a geometry is determined by its group of symmetries. (This fails for Riemannian geometry.) For example, affine geometry consists of the properties of space invariant under the group of affine transformsations, projective geometry is properties of projective space invariant under projective transformations, and so on. The group of affine transformations in n dimensicanons is a semidirect product $\mathbb{R}^n.GL_n(\mathbb{R})$. This can be identified with the subgroup of GL_{n+1} fixing a vector (sometimes called the mirabolic subgroup). For example, in 1-dimension we get a non-abelian 2-dimensional Lie group of transformations $x \mapsto ax + b$ with $a \neq 0$.