Solution to Math256a section IV.3 (H.Zhu, 1994)

3.1) One direction follows easily from 3.3.4. We show that D is not very ample when $deg(D) < 5$. Now suppose D is very ample, then $l(D) = l(D - P - Q) + 2 \geq 2$. Futhermore, if $l(D) = 2$, $dim|D| = 1$, thus $|D|$ defines an isomorphism from X to $\mathbb{P}^1$, which is obsurd. Thus we have $l(D) > 2$.

If $deg(D) \leq 1$, Since $l(D) \neq 0$, we may apply Ex. 1.5., $l(D) \leq deg(D)+1 \leq 2$, thus D is not very ample.

If $deg(D) = 2$, $l(D) = l(K - D) + 1$. Since $D \neq 0$, $l(K - D) < l(K) = 2$. Thus $l(D) \leq 2$. Contradiction.

If $deg(D) = 3$, then $l(K - D) = 0$. So $l(D) = 2$. Contradiction.

If $deg(D) = 4$, then $l(D) = 3$. By 3.2, we know that D is base point free. Thus $|D|$ defines a morphism from X to $\mathbb{P}^2$. But this is impossible since any plane curve has genus $(d - 1)(d - 2)/2$, which is never 2. Contradiction.

We conclude that $deg(D) \geq 5$.

3.2) (a) From I, Ex.7.2, $g(X) = 3$. It results in $l(K) = 3$ and $deg(K) = 4$. Denote $D =: X.L$. Recall Bezout's theorem from I, 7. so $deg(D) = 4$. Now claim that $l(D) \geq 3$. Since the line L on X is determined exactly by two points (not necessary distinct) so $dim|L| = 2$, i.e. $l(D) = 3$. (This may be rigorously proved by considering the possible linearly independent sets.) Then $l(K-D) = l(D)+g-deg(D)-1 = 1$. But $deg(K-D) = 0$ and $l(K-D) = deg(K-D)+1$, thus $K = D$ by Ex.1.5.

(b) Since D is an effective divisor of degree 2, $D = P_1+P_2$ for some two points on X (not necessary distinct). Suppose there is an effective divisor $Q_1 + Q_2$ such that $P_1 + P_2 \sim Q_1 + Q_2$. Since the line passing thru P_1 and P_2 intersects X at two other points P_3 and P_4. By (a) we have $K = P_1 + P_2 + P_3 + P_4$, so Q_1, Q_2, P_3, P_4 is collinear. Hence Q_1, Q_2 coincide with P_1, P_2. Thus $dim|D| = 0$.

(c) From Ex. 1.7.(a), $dim|K| = 1$. But we may pick an effective canonical divisor K such that $dim|K| = 0$ by (b). Thus X can not be a hyperelliptic curve.

3.3) It is clear that the second statement follows from the first one since K is not very ample on a hyperelliptic curve. (Cf. 5.2.) By II.Ex.8.4, $\omega_X \cong \mathcal{O}(\sum d_i - n - 1)$. Since the dimension of the global section of this invertible sheaf equals $g \geq 2$, ω_X has to be very ample. (Otherwise it has no global sections.) This is equivalent to saying that the canonical divisor K is very ample.

We showed in Ex.1.7 that any curve of genus 2 has to be a hyperelliptic curve, and its canonical divisor is not very ample. Thus it can not be a complete intersection in $\mathbb{P}^n$.

3.4) (a) Denote θ as the corresponding ring homomorphism. $deg(\theta) = d$. We know that the image of the d-uple embedding is $Z(ker(\theta))$. We may check that $ker(\theta)$ is generated by $x_{i+1}^2 - x_i x_{i+2}$, for $i = 0, \cdots, d-2$, and $x_0 x_d - x_1 x_{d-1}$.

(b) If i is the close immersion, denote $i^*(\mathcal{O}(1))$ as D. Because $dim|D| = n$ and $deg(D) = d$, we have $l(D) = n+1 \leq deg(D)+1 = d+1 \leq n+1$. Therefore $n = d$ and $g = 0$ by Ex..1.5. Consequently, $X \cong \mathbb{P}^1$. Since X does not lie in $\mathbb{P}^{n-1}$, the natural map $\Gamma(\mathbb{P}^n, \mathcal{O}(1)) \to \Gamma(X, i^*(\mathcal{O}(1))$ is injective. Thinking X

as $\mathbb{P}^1$, D corresponds to a $(n+1)$-dimension subspace $V \in \Gamma(\mathbb{P}^1, \mathcal{O}(n))$ hence they are equal since the later has dimension $n+1$. Therefore, X is indeed a rational normal curve. (see II. 7.8.1).

(c) It is clear this curve X can not be in $\mathbb{P}^1$. From (b), X is a plane curve of degree 2.

(d) Suppose X is not a plane cubic curve, we apply (b), have $X \subseteq \mathbb{P}^3 \backslash \mathbb{P}^2$, thus it is a rational normal curve of degree 3, which is indeed a twisted cubic.

3.6) (a) When $n \geq 4$, Ex.3.4 (b) implies that X is a rational normal curve. If X is a plane curve, $g(X) = (d-1)(d-2)/2 = 3$. Otherwise, $X \subseteq \mathbb{P}^3 \backslash \mathbb{P}^2$, we claim that $g = 0, 1$. Suppose $g = 2$, then Ex.3.1 shows that any divisor of degree 4 is not very ample, that is X can not be embedded to $\mathbb{P}^3$, which is absurd. If $g = 0$, then it is a rational quartic curve by II,7.8.6. g can not be 3 since X is not a plane curve. Thus g has to be 1.

3.7) Suppose C is a nonsingular curve which projects to the given curve X. We prove that $deg(C) = 4$ which will soon lead a contradiction with assertions in Ex.3.6. To prove our first claim, we carefully choose a suitable hyperplane H passing the projection point to cut C which intersects with $\mathbb{P}^2$ by a line L such that there is a 1-1 map from $C.H$ to $X.L$. We conclude that $deg(C) = 4$ by recalling Bezout's Theorem.

Since C has a node, it can not lie in case (1) or (2) in Ex.3.6. By Hurwitz's theorem, $g(C) \geq g(\tilde{X}) = 3 - 1$ from 3.11.1, thus $g(X) \neq 1$. Contradiction with Ex.3.6. Thus such C does not exist.

3.8) (a) By a simple calculation, the tangent vector is $(1, 0, 0)$ at each point. Pick an point $P = (x_0, y_0, z_0)$ on X, its tangent line is given by the intersection of two hyperplanes: $y = y_0$ and $z = z_0$. Writen in homogeneous polynomial, $y = y_0 w$ and $z = z_0 w$. Thus all tangent lines pass through the point at infinity $(1:0:0:0)$. There is one strange point on this curve.

(b) Note that when $char(k) = 0$, X has finitely many singular points. By choosing a proper projection, we may still project X in $\mathbb{P}^3$. Suppose P is a strange point on X. Choose an affine cover such that P is the infinity point on x-axis, and other relevant conditions in the proof of Theorem 3.9. The resulted morphism is ramified at all but finitely many points on X. The image is thus a point otherwise the map is inseparable which is not the case over a field of *char* 0. Hence X is the line $\mathbb{P}^1$.

3.9) Three points are collinear iff there is a multisecant line passing through them. A hyperpalne in $\mathbb{P}^3$ intersects X at exactly d points iff the hyperplane does not pass any tangent lines of X. Prop 3.5 showed that $dim(Tan(X)) \leq 2$. Also it is not hard to show that the dimension of the space of multisecant lines of X has dimension ≤ 1. Hence the union of these two spaces is a proper closed subspace of $\mathbb{P}^{3^*}$ which is of dimension 3. Therefore almost all hyperplanes intersect X in exactly d points.