Solution to Math256a section IV.1

1.1) Choose a positive integer n larger than $deg(K) = 2g - 2$, and g. By Riemann-Roch theorem, $l(nP) = n+1-g > 1$. Thus there exists a non-constant rational function f over X which has a pole at P of order $n > 0$, and regular everywhere else.

1.2) Induction on r. The case $r = 1$ follows from the previous exercise. Now assume there is a rational function f having poles at each of $P_1, \cdots, P_{r-1}$ of positive orders and regular everywhere else. Since f has no pole at P_r, we may let $n_r \geq 0$ be the coefficient of P_r in (f) (as a divisor). We may choose a rational function g which has a pole at P_r of order $> n_r$ and regular everywhere else (Cf. 1.1). Then $f \cdot g$ has poles precisely at $P_1, \cdots, P_r$ of positive orders.

1.5) Since D is effective, $|K - D| \subseteq |K|$. Therefore $l(K - D) \leq l(K)$. By Riemann-Roch theorem, $l(D) = l(K-D)-g+deg(D)+1 \leq l(K)-g+deg(D)+1 = deg(D)+1$, since $l(K) = g$. It follows that $dim(|D|) = l(D) - 1 \leq deg(D)$. Proof showes that the equality holds iff $l(K - D) = l(K) = g$. If $D = 0$, it is trivially true. If $g = 0$, $deg(K) = -2$, so $deg(K - D) < 0$. Hence $l(K - D) = 0$. It follows that $l(K - D) = l(K) = 0$.

Conversely, suppose $l(K - D) = l(K) = g$. Suppose $D \neq 0$. Let $P \in Supp(D)$. Then $|K - D| \subseteq |K - P| \subseteq |K|$, thus $l(K - D) \leq l(K - P) \leq l(K)$, and hence they are all equal. By Riemann-Roch, $l(P) = l(K - P) + 2 - g = 2$. Therefore there is a rational function f with one pole at P of order 1 and regualr everywhere else. This function defines an isomorphism from X to $\mathbb{P}^1$, thus $g(X) = g(\mathbb{P}^1) = 0$.

1.6) Let P be a point on X. By Riemann-Roch,

$$l((g + 1)P) = l(K - (g + 1)P) + (g + 1) + 1 - g \geq 2.$$

Thus there exists a rational function f with a pole at P of order $g + 1$ and regular everywhere else. This ration function induces a morphism $f : X \to \mathbb{P}^1$ by sending $(g+1)P$ to $\infty \in \mathbb{P}^1$. By II Prop. 6.9, $deg(f) = deg((g+1)P) = g+1$.

1.7) (a) It is clear that $deg(K) = 2g - 2 = 2$ and $dim(|K|) = l(K) - 1 = 1$. Suppose P is a base point of $|K|$, then $l(K - P) = l(K) = 2$ by definition. By Riemann-Roch, $l(P) = 2 + 2 - 2 = 2$. Thus there exist a non-constant rational function f with a pole at P of order 1 and regular everywhere else. As we did before, f defines an isomorphism from X to $\mathbb{P}^1$, contradiction since X has genus 2 not 0. Therefore $|K|$ has no base point. Alternatively, one may apply directly Prop 3.1 on page 307. By II, 7.8.1, there is a finite morphism $f : X \to \mathbb{P}^1$ with degree equal to $deg(K) = 2$. Therefore X must be a hyperelliptic curve.