Hartshorne, Chapter 1.4 Answers to exercises. REB 1994

4.1 If $f = g$ on $U \cap V$, then the function which is f on U and g on V is clearly regular. Therefore the union of all open sets on which f is represented by a regular function is the largest open set on which f is regular.

4.2 A map is regular if and only if it is regular in a neighborhood of each point, so the conclusion follows as in 4.1.

4.3a $f = x_1/x_0$ is defined in the set where $x_0 \neq 0$. This set is isomorphic to A^2, and f is just projection to the first coordinate.

4.3b ϕ is defined everywhere except the point $(1 : 0 : 0)$.

4.4a See exercise 3.1c.

4.4b The maps taking $t \in A^1$ to (t^3, t^2) and (x, y) to y/x (for $x \neq 0$) are inverse birational isomorphisms from the cuspidal curve to A^1.

4.4c The projection maps $(x : y : z)$ to $(x : y)$ if $(x : y : z) \neq (0 : 0 : 1)$. The inverse map from P^1 to Y takes $(x : y)$ to $((y^2 - x^2)x : (y^2 - x^2)y : x^3)$ for $(x : y) \neq (1 : \pm 1)$.

4.5 The subvariety of Q given by $w \neq 0$ is isomorphic to A^2 by $(w : x : y : z) \to (x/w, y/w)$, $(x, y) \to (1 : x : y : xy)$. Thererfore Q is birational to $Q - \{w = 0\}$, which is isomorphic to A^2, which is birational to P^2. Q is isomorphic to $P^1 \times P^1$, which is not isomorphic to P^2 as it contains 2 closed 1-dimensional subvarieties that do not intersect.

4.6ab Put $U = V = \{(x : y : z)|xyz \neq 0\}$. Φ clearly maps U to V, and ϕ^2 maps $(x : y : z)$ to $(ax : xy : az) = (x : y : z)$ where $a = xyz$, so ϕ^2 is the identity map.

4.6c $\phi = \phi^{-1}$ is defined everywhere on P^2 except at the points $(1 : 0 : 0)$, $(0 : 1 : 0)$, $(0 : 0 : 1)$. Remark: The group of birational transformations of P^2 is generated by quadratic transformations (or by one quadratic transformation and $PGL_3(k)$) and very little about it seems to be known beyond the fact that it is very large.

4.7 We can assume that X and Y are closed subsets of A^n, and $P = Q = 0$. If f is a homomorphism from $O_{Q,Y}$ to $O_{P,X}$ then define a map g from an open subset of X to Y by

$$g(x_1, \ldots, x_n) = (f(y_1)(x_1, \ldots, x_n), f(y_2)(x_1, \ldots), \ldots)$$

where y_i is the i'th coordinate function on A^n. This is defined on the open set where all the $f(y_i)$'s are defined. Likewise we can define a similar map from an open set of Y to X, and the composition of these two maps is the identity wherever it is defined. Therefore there is an isomorphism from an open set of X to an open set of Y taking P to Q.

4.8a Clearly the cardinality of P^n is at most $(n + 1)card(k)^n$ which is the cardinality of k. To prove the other inequality we can assume that X is contained in A^n. If the possible values of any coordinate $x_1, \ldots, x_n$ are finite, the X consists of a finite number of points, so we can assume that one coordinate, say x_1, takes on an infinite number of values. By elimination theory the condition for a point with a given value of x_1 to exist on X is given by a finite number of equalities and inequalities in x_1. Therefore the possible values of x_1 are either a finite set or the complement of a finite set in k. But we know the number of possible values of x_1 is infinite, so the number of values is the cardinality of k minus a finite number, which is the cardinality of k.

4.8b Any two curves have the same cardinality and the finite complement topology, and so are homeomorphic.

4.9 We can assume that X is affine and is contained in A^n, the set of points in P^n with first $x_0 \neq 0$. The field of fractions $k(X)$ is generated by $x_1, \ldots, x_n$, so we can assume that $x_1, \ldots, x_r$ is a separating transcendence basis for $k(X)/k$ by 4.7A and 4.8A, and $k(X)$ is generated by $a_{r+1}x_{r+1} + \cdots + a_nx_n$ for some a_i's in k, by 4.6A. As $r \leq n - 2$ we can find a form $b_{r+1}x_{r+1} + \cdots + b_nx_n$ not proportional to $a_{r+1}x_{r+1} + \cdots + a_nx_n$. Choose any point at infinity not in this plane or in $\bar{X}$. Then the projection from this point to the plane maps $k(hyperplane)$ onto $k(X)$, so it is an isomorphism from the function field of the image of X to $k(X)$, and therefore a birational isomorphism.

4.10 If $(x, y, w : z) \in A^2 \times P^2$ is in $\phi^{-1}(Y)-$(exceptional curve) then $y^2 = x^3$, $xz = yw$, so $x^2(z^2 - xw^2) = 0$, so $z^2 - xw^2 = 0$. Therefore the only possibility for this point to lie on the exceptional curve $x = y = 0$ is $(0, 0, 1 : 0)$. If $w = 0$ then $x = 0$ which is not possible, so we can define the map f from $\bar{Y}$ to A^1 by $f(x, y, w : z) = z/w$. The inverse takes t to $(t^2, t^3, 1 : t)$, so $\bar{Y}$ is isomorphic to A^1.