Hartshorne, Chapter 1.3 Answers to exercises. REB 1994

3.1a Follows from exercise 1.1 as 2 affine varieties are isomorphic if and only if their coordinate rings are.

3.1b The coordinate ring of any proper subset of A^1 has invertible elements not in k and o is not isomorphic to the coordinate ring of A^1.

3.1c The aut group of P^2 acts transitively on sets of 3 points not on a line, so we can assume the conic contains $(0:0:1)$, $(0:1:0)$, and $(1:0:0)$, i.e., it is of the form $axy+byz+czx=0$ for some a,b,c, which are nonzero as otherwise the conic would be a union of two lines. We can multiply x, y, and z by constants to make a, b, and c all equal to 1, so we can assume the conic is $xy+yz+zx=0$, and in particular all conics are isomorphic. Hence we only have to show 1 conic is isomorphic to P^1, e.g., the image of P^1 under the 2-uple embedding.

3.1d Any 2 1-dimensional closed subsets of P^2 intersect (see ex. 3.7a), but A^2 does not have this property.

3.1e By theorem 3.4 the regular functions on a projective variety is the ring k, which is only possible for an affine variety if it is a point.

3.2a If ϕ had an inverse, this would give a polynomial $f(x,y)$ such that $f(t^2,t^3)=t$, which is impossible.

3.2b ϕ is 1:1 because if $x^p=y^p$ in characteristic p then $(x-y)^p=0$ so $x=y$. It has no inverse because there is no polynomial f with $f(t^p)=t$.

3.3a If f is a regular function defined on a neighborhood V of $\phi(p)$ then $f\circ\phi$ is a regular function on the neighborhood $\phi^{-1}(V)$ of p, This gives a map from $O_{\phi(p),Y}$ to $O_{p,X}$ which is a homomorphism.

3.3b We have to show that if V is an open set in X, and f is regular on V, then $f\circ\phi^{-1}$ is regular on $\phi(V)$. If $\phi(p)\in\phi(V)$ then $f\in O_{p,X}$, so ϕ_p^{-1*} maps f to an element of $O_{\phi(p),y}$, so $f\circ\phi^{-1}$ is regular near $\phi(p)$, so it is regular on $\phi(V)$.

3.3c If $\phi_p^*(f)=0$ then f vanishes on $\phi(X)\cap V$ which is a dense subset of V. As f is continuous and vanishes on a dense subset, it must be 0. Therefore ϕ_p^* is injective.

3.4 It is enough to show that ϕ^{-1} is regular near $\phi(1:x_1:\cdots:x_n)$, where ϕ is the d-uple embedding. But near this point ϕ^{-1} takes $(m_0:\cdots:m_N)$ to $(m_{i_0}:\cdots:m_{i_n})$ where m_{i_k} is the coordinate corresponding to the monomial $x_0^{d-1}x_k$, and this is a regular map.

3.5 Identify P^n with its image under the d-uple embedding. Then H is the intersection of a hyperplane in P^N with P^n, so P^n-H is a closed subset of $P^N-H=A^N$ and is therefore an affine variety.

3.6 Any regular function on X has the form $f(x,y)/g(x,y)$ where f and g are coprime. The curves of f and g only intersect in a finite number of points and g can only vanish at $(0,0)$ or where $f=0$, so g has only a finite number of zeros and must therefore be constant. Hence $O(X)=k[x,y]$. Therefore the map from X to A^2 is an isomorphism of their coordinate rings, so if X was affine it would be an isomorphism of varieties, which it obviously is not as is is not surjective on points.

3.7b Suppose $Y\cap H=\phi$. Then Y is a closed subset of an affine variety P^n-H and therefore a finite set of points, as any projective subset of an affine variety is finite.

3.8 Any regular function on P^n-H_i is of the form $f_i(x_0,\dots,x_n)/x_i^{d_i}$ where d_i is the degree of the homogeneous polynomial f_i. Hence for a function to be regular except on $H_i\cup H_j$ we would have $f_ix_j^{d_j}=f_jx_i^{d_i}$ for some f_i, f_j. But this implies $f_i=x_i^{d_i}$, so the function must be constant.

3.9 $S(X)$ is the polynomial ring $k[X_0,X_1]$, but $S(Y)$ is the subring $k[X_0^2,X_0,X_1,X_1^2]$ of $k[X_0,X_1,X_2]$, which is not a graded polynomial ring in 2 variables (as the space of elements of the smallest nonzero degree is 3 dimensional).

3.10 For any point $x\in X'$ there is an affine neighborhood U of x in X and a regular function f from U to Y with $\phi|U=f$. Therefore f is a regular function from the neighborhood $U\cap X'$ of x to Y and therefore to Y'. Hence ϕ is regular near each point of X' and is therefore regular.

3.11 We can assume that X is affine as the irreducible varieties of X containing P are just the closures of the irreducible varieties containing P of any affine neighborhood of P. But then the varieties containing P just correspond to the prime ideals of $A(X)$ contained in the maximal ideal M of P, which correspond to the prime ideals of the ring $A(X)$ localized at M, which are the prime ideals of the local ring O_P.

3.12 By exercise 2.6 there is an affine neighborhood Y of P with $\dim(Y)=\dim(X)$. But $O_{P,X}=O_{P,Y}$ so $\dim(X)=\dim(Y)=\dim(O_{P,Y})$ (by 3.2c) $=\dim(O_{P,X})$.

3.13 $O_{Y,X}$ is clearly a ring. Put I = image of set of pairs $\{U,f\}$, f regular on U, with $f=0$ on $U\cap Y$. Then I is the unique maximal ideal, because if $\{V,g\}$ is not in I then it has an inverse $\{W,1/g\}$ where $W=V\cap$(set where $g\neq 0$), as $W\cap Y\neq 0$. The residue field is obviously $K(Y)$. To prove the result about dimensions, we can assume X affine. Put $B = A(X)$, p=functions on X vanishing on Y. Then by 1.8A, $\text{height}(p) + \dim(B/p) = \dim(B)$. But $\dim(B) = \dim(X)$ and $\dim(B/p) = \dim(Y)$ and height of p in B = height of maximal ideal of $O_{Y,X}$ = dimension of $O_{X,Y}$. Hence $\dim(O_{X,Y} + \dim(Y) = \dim(X)$.

3.14a We can assume that P^n is the set where $x_0 \neq 0$, and p is the point $(1 : 0 : \cdots : 0)$. If $x = (x_0 : \cdots : x_n) \in P^{n+1} - P$, then $x_i \neq 0$ for some $i > 0$. Therefore the line containing P and x meets P^n in $(0 : x_1 : \cdots : x_n)$, which is a morphism in the neighborhood $x_i \neq 0$ of x. Therefore ϕ is a morphism.

3.14b The projection maps (t^3, t^2u, tu^2, u^3) to $(t^3, t^2u, u^3) \in P^2$. It is easy to check that the image is the whole of the variety given by the equation $x_1^3 = x_2x_0^2$. For $x_2 \neq 0$ this is the same as the variety given by $y^3 = x^2$ which has a cusp at $(0, 0)$, i.e., the image has a cusp at $(0, 0, 1)$.