Hartshorne, Chapter 1 Answers to exercises. REB 1994

2.1 a is homogeneous and so defines a cone in A^{n+1}. f vanishes on all the elements of this cone (including 0 as f has positive degree) so $f^q \in a$ for some $q > 0$ by the usual Nullstellensatz.

2.2 (iii) implies (i) is trivial as $x_i^d \in S_d$. Proof that (i) implies (ii): If $Z(a)$ is empty, then in A^{n+1}, $Z(a)$ must be empty or $(0, \ldots, 0)$, so $\sqrt{a}$ must be S or $\oplus_{d>0} S_d$. Proof that (ii) implies (iii): $\sqrt{a}$ contains x_i, so there is some m with $x_i^m \in a$ for all i, so a contains $S_{m(n+1)}$ as any monomial of degree $m(n+1)$ must have x_i^m as a factor for some i.

2.3 (a),(b),(c),(e) are trivial. For (d), clearly $I(Z(a))$ contains $\sqrt{a}$. As $Z(a)$ is nonempty, any nonzero homogeneous polynomial vanishing on it must have positive degree. By 2.1, this implies that $f^q \in a$. Therefore $I(Z(a))$ is contained in $\sqrt{a}$ as it is a homogeneous ideal.

2.4a Follows from 2.3d,e, and 2.2.

2.4b If $Y = Y_1 \cup Y_2$, then $I(Y) = I(Y_1) \cap I(Y_2) \supset I(Y_1)I(Y_2)$. Therefore if $I(Y)$ is prime, $I(Y)$ must be either $I(Y_1)$ or $I(Y_2)$, so Y is Y_1 or Y_2. On the other hand if Y is not prime, then $ab \in I(Y)$, with $a \notin I(Y)$, $b \notin I(Y)$. Therefore Y is the union of the proper subsets $Y \cap Z(a)$, $Y \cap Z(b)$ and is therefore not irreducible.

2.4c $I(P^n) = 0$ which is a prime ideal.

2.5a P^n can be covered by $n + 1$ copies of A^n which is Noetherian.

2.5b See proposition 1.5 and part (a) of this question.

2.6 S(Y) is the coordinate ring of the cone in A^{n+1} corresponding to Y (assuming Y is nonempty). $S(Y)_{x_i}$ is the coordinate ring of the cone $Y - (x_i = 0)$ if x_i is not identically 0 on Y, i.e., Y_i is nonempty. Therefore the homogeneous part of degree 0 of $S(Y)_{x_i}$ is the coordinate ring of the cone with $x_i = 0$, which is isomorphic to Y_i, and therefore $S(Y)_{x_i} = A(Y_i)[x_i, 1/x_i]$ as every element of $S(Y)_{x_i}$ is the sum of monomials of the form ($x_i^{\pm n} \times$ element of degree 0). Therefore $Tr.deg.(S(Y)_{x_i}) = Tr.deg.(A(Y_i) + 1) = Tr.deg.S(Y)$. Therefore $\dim(S(Y)) = 1 + \dim(Y_i)$ whenever $Y_i \neq 0$. The Y_i's cover Y, so $\dim(Y) = \sup(\dim(Y_i))$, so $\dim(S(Y) = 1 + \dim(Y)$.

2.7a P^n is covered by $n + 1$ open copies of A^n, so $\dim(P^n) = \sup(\dim(A^n)) = n$.

2.7b Y is contained in P^n, and therefore covered by $n+1$ copies of A^n. In each copy A_i of A^n, $\overline{Y \cup A_i} = \overline{Y} \cup A_i$ as A_i is open. Hence $\dim(Y \cap A_i) = \dim(\overline{Y \cap A_i}) = \dim(\overline{Y} \cap A_i)$, and therefore $\dim(Y) = \sup(\dim(Y \cap A_i) = \sup \dim(\overline{Y} \cap A_i) = \dim(\overline{Y})$.

2.8 If f is any homogeneous polynomial of positive degree then the zero set of f has dimension $n - 1$ as it has this dimension on some affine subsets and is a proper closed subset of P^n. Also f is irreducible, so the homogeneous ideal generated by it is prime (as rings of polynomials are U.F.D.'s so irreducibles are primes) so its variety is irreducible. Conversely if Y is any proper closed subset of P^n then there is some homogeneous polynomial f vanishing on Y which we can assume to be irreducible because Y is irreducible (so some factor of f must also vanish on Y if f is not irreducible). Then the zero set of f is an irreducible $n - 1$ dimensional closed subset of P^n containing the $n - 1$ dimensional closed subset Y, and so must be equal to Y (because any proper closed subset of an irreducible topological space has smaller dimension).

2.9a $\beta g(x_0, \ldots, x_n) = x_0^d g(x_1/x_0, \ldots, x_n/x_0)$ if g is of degree d. If g vanishes on Y then βg vanishes on $\bar{Y}$, so $I(\bar{Y}) \supseteq \beta(I(Y))$. If h vanishes on $\bar{Y}$ then we can assume h is homogeneous. If $g(x_1, \ldots, x_n) = h(1, x_1, \ldots, x_n)$, then $h = \beta g$, so $I(\bar{Y})$ is generated by $\beta(I(Y))$

2.9b $\{(t, t^2, t^3)\} = Y$, and $I(Y) = (x_2 - x_1^2, x_3 - x_1^3)$. $\beta(x_2 - x_1^2) = x_0 x_2 - x_1^2$ and $\beta(x_3 - x_1^3) = x_0^2 x_3 - x_1^3$. But $I(\bar{Y})$ contains $x_1 x_3 - x_2^2$ which is not contained in $(\beta(x_2 - x_1^2), \beta(x_3 - x_1^3))$.

2.10a Obvious.

2.10b They have the same ideal, which is prime if and only if they are irreducible.

2.10c See 2.6.

2.11a $I(Y)$ is generated by linear polynomials $\{p_i\}$ if and only if Y is the intersections of the hyperplanes $\{p_i = 0\}$.

2.11b Any hyperplane in P^n is a copy of P^{n-1}, and the intersection of any other hyperplane of P^n with this P^{n-1} is a hyperplane of the P^{n-1}. Therefore any r-dimensional linear variety in P^n is the intersection of $n - r$ hyperplanes and not the intersection of $n - r - 1$ hyperplanes. Therefore its ideal is minimally generated by $n - r$ linear polynomials.

2.11c Y is the intersection of $n-r$ hyperplanes and Z is the intersection of $n-s$ hyperplanes, so $Y \cap Z$ is the intersection of $2n-r-s$ hyperplanes, which has dimension at least $n-(2n-r-s) = r+s-n$. In particular it is nonempty if $r+s \geq n$.

2.12a θ maps $k[y_0, \ldots, y_N]$ to an integral domain, so its kernel is a prime ideal. If $f \in k[y_0, \ldots, y_N]$, $f = f_0 + f_i + \cdots$ with f_i of degree i, then $\theta(f_i)$ has degree di, so $\theta(f) = 0$ if and only if $\theta(f_i) = 0$ for all i, and therefore the kernel is also a homogeneous ideal.

2.12b If $f \in Ker(\theta)$ then $f(M_0, \ldots, M_n) = 0$. Hence f vanishes on any point $(M_0(a), \ldots, M_n(a))$, so $Im(\rho_d) \subseteq Z(a)$. This proves the easy half. Any monomial raised to the power of d is a product of monomials of the form x_i^d. Choose any point $(m_0, \ldots, m_N) \in Z(a)$. Some m_i is nonzero and $m_i^d = \prod_N m_{j_n}$ where each m_{j_N} corresponds to some monomial x_i^d, hence some m_i corresponding to a monomial x_i^d is nonzero; say $i = 0$. If $m_{i_1}, \ldots, m_{i_n}$ correspond to $x_0^{d-1}x_1, \ldots, x_0^{d-1}x_n$ then put $x_0 = 1$, $x_k = m_{i_k}/m_0$, and try to use this to define a map to P^n on the set with $m_0 \neq 0$. We have to show that $m_0 M_i(1, x_1, \ldots, x_n) = m_i$ (where m_0 corresponds to x_0^n), i.e., that $m_0 M_i(1, m_{i_1}/m_0, \ldots, m_{i_n}/m_0) = m_i$. But this is true because $x_0^d M_i(1, x_1/x_0, \ldots, x_n/x_0) = M_i(x_0, \ldots, x_n)$, and therefore $(m_0, \ldots, m_N)$ is the image of $(x_0, \ldots, x_n)$. Hence $Im(\rho_d) \supseteq Z(a)$.

2.12c ρ_d is continuous and bijective from P^n to $Z(a)$. To show that it is a homeomorphism it is sufficient to show that its inverse is continuous on any open set of $Z(a)$ of the form $m_i \neq 0$ (notation as above) because these open sets cover $Z(a)$. But this follows from the construction of this inverse above.

2.12d The 3-tuple embedding of P^1 into P^3 maps $(x_0 : x_1)$ to $(x_0^3 : x_0^2 x_1 : x_0 x_1^2 : x_1^3)$ which is the projective closure of $\{(x_1, x_1^2, x_1^3)\}$ in P^3, i.e., the twisted cubic curve.

2.13 The map is given by $(x_0 : x_1 : x_2) \to (x_0^2 : x_1^2 : x_2^2 : x_0x_1 : x_1x_2 : x_2x_0)$. Any curve in P^2 is defined by some polynomial $f(x_0, x_1, x_2) = 0$, f homogeneous, and therefore also by the polynomial $f(x_0, x_1, x_2)^2 = g(x_0^2, x_1^2, x_2^2, x_0x_1, x_1x_2, x_2x_0)$ for some polynomial g. Then some factor of this polynomial g defines a suitable hypersurface containing the image of the curve Z. (This assumes that P^2 is isomorphic to its image which is easy to check (see 2.14 below) once one has defined isomorphisms of varieties, so that curves in the image of P^2 correspond to curves in P^2.)

2.14 The image of ψ is the set Y defined by the equations of the form $x_{ab}x_{cd} = x_{ac}x_{bd}$. Proof: the image is clearly contained in Y. Conversely if $(x_{00} : x_{10} : \cdots : x_{rs}) \in Y$ the we may assume that x_{00} (say) is nonzero. But then the point is the image of $(x_{00} : x_{10} : \cdots : x_{r0}) \times (x_{00} : x_{01} : \cdots : x_{0s}) \in P^r \times P^s$.

2.15a $(a_0 : a_1) \times (b_0 : b_1) = (a_0b_0 : a_0b_1 : a_1b_0 : a_1b_1) = (w : x : y : z)$, and the image of $P^1 \times P^1$ is then the subvariety $xt - zw = 0$ as in 2.14.

2.15b Q is isomorphic to $P^1 \times P^1$, so we can take the two families of lines to correspond to point$\times$line and line$\times$point. (It is easy to check that these are lines in $Q \subset P^3$; for example the image of $(a_0 : a_1) \times P^1$ is the set of points $(w : x : y : z) \in P^3$ with $a_1 w = a_0 y$, $a_1 x = a_0 z$.)

2.15c The closed subset $x = y$ of Q is not one of these lines.

2.16a $x^2 = yw$, $xy = zw$, so $y^2 w = xzw$, so $w = 0$ or $y^2 = xz$. Hence $Q_1 \cap Q_2$ is the intersection of the line $w = x = 0$ and the twisted cubic $x^2 = yw, xy = zw, y^2 = xz$.

2.16b $L \cap C$ is the point $P = (0 : 0 : 1)$, so $I(P) = (x, y)$, but $I(L) + I(C) = (x^2, y) \neq (x, y)$.

2.17a By problem 1.8, the intersection of q hypersurfaces has dimension at least $n - q$. If a can be generated by q elements then $Z(y)$ is the intersection of q hypersurfaces and therefore has dimension at least $n - q$ (using problem 2.8).

2.17b If $I(Y)$ can be generated by r elements then Y is the intersection of their hypersurfaces.

2.17c Y is the intersection of $H_1 = Z(x^2 - wy)$ and $H_2 = Z(y^3 + wz^2 - 2xyz)$ as $(xy - wz)^2 = w(y^3 + wz^2 - 2xyz) + y^2(x^2 - wy)$ and $(y^2 - xz)^2 = y(y^3 + wz^2 - 2xyz) + z^2(x^2 - wy)$, and $y^3 = wz^2 - 2xyz = y(y^2 - xz) + z(wz - xy)$. On the other hand $I(Y)$ has no homogeneous elements of degree 0 or 1 and the space of homogeneous elements of degree 2 is 3 dimensional, so any set of generators must have at least 3 elements.

2.17d Still an unsolved problem (as far as I know).