Homework 6.

14.2 $-i$ has absolute value 1 and argument $-\pi/2$. So $\log(-i) = \log(1) + i(2n - 1/2)\pi = i(2n - 1/2)\pi$ for integers n.

14.6 $(1-i)/\sqrt{2}$ has absolute value 1 and argument $-\pi/4$ So $\log((1-i)/\sqrt{2}) = i(2n - 1/4)\pi$ for integers n.

14.9 $(-1)^i = \exp(i\log(-1))$. We know $\log(-1) = (2n+1)\pi i$ for any integer n, so the values of $(-1)^i$ are $\exp(-(2n+1)\pi)$ for integers n. For example, one value is $e^\pi = 23 \cdot 14 \ldots$

14.19 $\cos(\pi + i\log(2)) = -\cos(i\log(2)) = -\cosh(\log(2)) = -(e^{\log(2)} + e^{-\log(2)})/2 = -(2 + 1/2)/2 = -5/4$.

14.24(b) i^{-1} has only one value $-i$. But $i^i = \exp(i\log(i)) = \exp(i(2n + 1/2)i\pi) = \exp(-(2n + 1/2)\pi)$, so $\log(i^i) = -(2n+1/2)\pi + 2m\pi i$, so $(i^i)^i = \exp((-(2n+1/2)\pi + 2m\pi i)i) = \exp(2m\pi - \pi i/2) = -i\exp(2m\pi)$ for any integer m.

15.1 If $z = \arcsin(2)$ then $2 = \sin(z) = (e^{iz} - e^{-iz})/2i$. Put $a = e^{iz}$. Then $a - a^{-1} = 4i$, so $a^2 - 4ia - 1 = 0$, so $a = 2i \pm \sqrt{-3} = (2 \pm \sqrt{3})i$. So $iz = \log(a) = \log(2 \pm \sqrt{3}) + (2n+1)\pi i/2$. So $z = -i\log(2 \pm \sqrt{3}) + (2n+1)\pi/2$.

15.2 If $z = \arctan(2i)$ then $2i = \tan(z) = (a - a^{-1})/i(a + a^{-1})$ where $a = e^{iz}$. Solving for a gives $a^2 = -1/3$, so $a = \pm i/\sqrt{3}$. Therefore $iz = \log(1/\sqrt{3}) + i\pi(n + 1/2)$ so $z = \pi(n + 1/2) + \log(3)/2$.

15.3 We want to solve $1/2 = \cosh(z) = (e^z + e^{-z})/2$, so $e^z = 1/2 \pm \sqrt{3}i/2$, so $z = \pi i(2n \pm 1/3)$.

15.17 We want to show that $\tan(z) = \pm i$ has no solutions. This equation is equivalent to $(a - 1/a)/(a + 1/a) = \pm 1$ where $a = e^{iz}$. But this equation for a has no solutions in a for non-zero complex numbers. (Note that a must be non-zero as it is for the form e^{iz}.)

16.11 Put $z = e^{i\theta}$. Then $z + z^3 + z^5 + \cdots + z^{2n-1} = z(1 - z^{2n})/(1 - z^2)$ (geometric series) $= (1 - z^{2n})/(z - z - 1) = i(1 - \cos(2n\theta) - i\sin(2n\theta))/2\sin(\theta)$. Equating real and imaginary parts shows that the sum of the sin series in the question is $(1 - \cos(2n\theta))/2\sin(\theta)$, and the sum of the cos series is $\sin(2n\theta)/2\sin(\theta)$. This is equivalent to the answer given because $1 - \cos(2n\theta) = 2\sin^2(n\theta)$.

17.1 $(1+i)/(1-i) = i$, so its 2718'th power is $i^{2718} = (-1)^{1359} = -1$.

17.7 $(-i)^i = \exp(i\log(-i)) = \exp(i(2n - 1/2)\pi i) = \exp((1/2 - 2n)\pi)$.

17.14 Circle, center $2i$, radius 1.

17.17 Done in class.

17.19 Done in class.

17.23 $\cos(iz) = (\exp(i^2 z) + \exp(-i^2 z))/2 = (\exp(-z) + \exp(z))/2 = \cosh(z)$.

17.24 Similar to 17.23.

17.25a $\overline{\cos(z)} = \overline{(\exp(iz) + \exp(-iz))/2} = \overline{\exp(iz)}/2 + \overline{\exp(-iz)}/2 = \exp(i\bar{z})/2 + \exp(-i\bar{z})/2 = \cos(\bar{z})$.

17.25b Yes.

17.25c No; try $z = 1$.

17.25d Follows because $\overline{a_n z^n} = \overline{a_n}\bar{z}^n = a\bar{z}^n$ if a is real.

17.30 $\exp((1+i)x) = \sum(1+i)^n x^n/n!$. Also $(1+i)^n = 2^{n/2}e^{n\pi/4}$. If n is $2 + 4m$ for some integer m then $e^{n\pi/4}$ is $\pm i$ and so its real part is 0. We know that $e^x\cos(x)$ is the real part of $\sum(1+i)^n x^n/n!$, so the coefficient of x^n vanishes if n is of the form $2 + 4m$. $e^x\cos(x) = \sum x^n((1+i)^n + (1-i)^n)/n!$. The result for $e^x\sin(x)$ is similar: this time the coefficients of x^0, x^4, x^8, and so on vanish.

17.32 This series is $\exp(1 + i\pi) = \exp(1)\exp(i\pi) = \exp(1) = -e$.