Math 113 Final, Friday December 10, 8:10–11:00.

Please make sure that your name is on everything you hand in.
This is a "closed book" exam. (Calculators are not allowed.)
Answer as many questions as you can.
All ten questions have about the same number of marks.

1. Find integers m and n such that $21m + 50n = 1$.

2. Let F be a field. Prove that there is an infinite number of irreducible polynomials in $F[x]$.

3. Let G be the dihedral group of order 10 consisting of all symmetries of a regular pentagon (with 5 sides). Find the conjugacy classes in G.

4. Show that there is an infinite number of integers a such that $x^5 + 6x + a$ is irreducible over the rational numbers. Show that there is an infinite number of integers a such that this polynomial is reducible over the rational numbers.

5. If K is a field containing a field F such that the degree $[K : F]$ is odd, show that K has no subfield of degree 2 over F. Show that $\sqrt{2}$ is not contained in any extension of the rational numbers of odd degree.

6. Which of the following polynomials are irreducible? Give reasons for your answers.
   (a) $x^3 - x - 1$ over the field F_3 with 3 elements.
   (b) $x^4 + 6x - 4$ over the field Q of rational numbers.
   (c) $x^4 + 8x^2 + 6$ over the field Q of rational numbers.

7. Find a solution in the field F_5 with 5 elements of the equations

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 3 \\ x & - & y & + & z & = & 1 \\ 2x & + & y & - & z & = & 2 \end{array}$$

8. Let F be a field of characteristic $p \neq 0$ and let n be a positive integer. Show that $(a + b)^p = a^p + b^p$ for all $a, b \in F$. Show that $(a + b)^{p^n} = a^{p^n} + b^{p^n}$ for all $a, b \in F$. Show that the set of all roots of $x^{p^n} - x = 0$ in F is a finite subfield of F.

9. Show that each of the following complex numbers are algebraic numbers, and determine their degrees over the rational numbers Q.
   (a) $\sqrt{3} + \sqrt{7}$
   (b) $\cos(2\pi/23) + i \sin(2\pi/23)$
   (c) $\frac{(1+i)}{\sqrt{2}}$

10. Prove that the number of elements of any finite field is a power of a prime number.