Math 53 - Gaussian Integral Problem

We start with two random variables x and y. They both have mean of 0, and have standard deviations σ_1 and σ_2 respectively. We will let $f(x)$ and $g(y)$ be their probability density functions:

$$f(x) = \frac{1}{\sigma_1\sqrt{\pi}} e^{-x^2/\sigma_1^2} \qquad g(y) = \frac{1}{\sigma_2\sqrt{\pi}} e^{-y^2/\sigma_2^2}$$

Now we'll define a new random variable $X := x + y$. We'll also assume X is also normally distributed. (It is.) So it has a probability density function

$$p(X) = \frac{1}{\Sigma\sqrt{\pi}} e^{-X^2/\Sigma^2}$$

for some standard deviation Σ. The goal is to find Σ in terms of σ_1 and σ_2.

We'll start by considering the probability that for some value S, $0 \le X \le S$. Since $X = x + y$, this happens when $0 \le x + y \le S$. So we get:

$$\int_0^S p(X)dx = Pr\left[0 \le X \le S\right] = Pr\left[0 \le x + y \le S\right] = \iint\limits_{0 \le x+y \le S} f(x)g(y)dA = \int_{-\infty}^{\infty}\int_{-x}^{S-x} f(x)g(y)dy\, dx$$

Now we differentiate both ends with respect to S and use the fundamental theorem of calculus:

$$p(S) = \frac{d}{dS}\int_{-\infty}^{\infty}\int_{-x}^{S-x} f(x)g(y)dy\, dx = \int_{-\infty}^{\infty}\frac{d}{dS}\int_{-x}^{S-x} f(x)g(y)dy\, dx = \int_{-\infty}^{\infty} f(x)g(S-x)dx$$

Now we will substitute in $S = 0$ to get:

$$\frac{1}{\Sigma\sqrt{\pi}} = p(0) = \int_{-\infty}^{\infty} f(x)g(-x)dx = \int_{-\infty}^{\infty} \frac{1}{\sigma_1\sqrt{\pi}} e^{-x^2/\sigma_1^2} \frac{1}{\sigma_2\sqrt{\pi}} e^{-(-x)^2/\sigma_2^2}$$

$$= \frac{1}{\sigma_1\sigma_2\pi}\int_{-\infty}^{\infty} e^{-\left(x\sqrt{\sigma_1^{-2}+\sigma_2^{-2}}\right)^2} = \frac{1}{\sigma_1\sigma_2\pi}\frac{1}{\sqrt{\sigma_1^{-2}+\sigma_2^{-2}}}\sqrt{\pi} = \frac{1}{\sqrt{\sigma_1^2+\sigma_2^2}\sqrt{\pi}}$$

So $\Sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$.