Math 113 Homework 7 Solutions

(1) Find all the ideals of $\mathbb{Z}_{12}$. Which ideals are prime? Which are maximal?

$\{0\}$, $\mathbb{Z}_{12}$, $2\mathbb{Z}_{12}$, $3\mathbb{Z}_{12}$, $4\mathbb{Z}_{12}$, and $6\mathbb{Z}_{12}$. The prime ones are $\mathbb{Z}_{12}$, $2\mathbb{Z}_{12}$, and $3\mathbb{Z}_{12}$. The maximal ones are $2\mathbb{Z}_{12}$, and $3\mathbb{Z}_{12}$.

(2) Given two rings, R_1 and R_2, we can define a direct product $R_1 \times R_2$ of order pairs (r_1, r_2) with $r_1 \in R_1$ and $r_2 \in R_2$.

(a) How should we define addition and multiplication in $R_1 \times R_2$? What are 0 and 1 in $R_1 \times R_2$? Prove that your answer is actually a ring.
$(r_1, r_2) + (s_1, s_2) = (r_1 + s_1, r_2 + s_2)$. $(r_1, r_2) \cdot (s_1, s_2) = (r_1 \cdot s_1, r_2 \cdot s_2)$. *1 is* $(1_{R_1}, 1_{R_2})$. *0 is* $(0_{R_1}, 0_{R_2})$. *Check the ring axioms.*

(b) Show that all ideals in $\mathbb{Z} \times \mathbb{Z}$ are principal.

Given an ideal I in $\mathbb{Z} \times \mathbb{Z}$, let $i_1 = min\{a > 0 : \text{for some } b, (a, b) \in I\}$ and $i_2 = min\{b > 0 : \text{for some } a, (a, b) \in I\}$. Can then prove that $I = ((i_1, i_2))$.

(3) Do exercise 14.16 in Judson

Suppose R is a field. Then $0 \neq 1$, so $\{0\} \neq R$ and R has at least two ideals. Suppose $I \neq \{0\}$ is an ideal of R, we will prove it equals R. Let i be a nonzero element of I and let a be any element of R. Then $\left(a\frac{1}{i}\right) i = a$, so $a \in I$. Thus $I = R$.

(4) Do exercise 14.35 in Judson

$\mathbb{Z}_{(p)}$ is a subset of $\mathbb{Q}$. We need to check it is a subring. Clearly 1 is in $\mathbb{Z}_{(p)}$. If $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$ are in $\mathbb{Z}_{(p)}$. Then since p does not divide b_1 or b_2, also it does not divide $b_1 b_2$. So $\frac{a_1 a_2}{b_1 b_2} = \frac{a_1}{b_1} \cdot \frac{a_2}{b_2}$ and $\frac{a_1 b_2 - a_2 b_1}{b_1 b_2} = \frac{a_1}{b_1} - \frac{a_2}{b_2}$ are in $\mathbb{Z}_{(p)}$. So it is a subring.

(5) Do exercise 14.41 in Judson

(a) *Since $R = I + J$, write $r = i + j$ and $s = i' + j'$ for some $i, i' \in I$ and $j, j' \in J$. Let x be $i' + j$. Then $x - r = i' - i \in I$ and $x - s = j - j' \in J$.*

(b) *If x_1 and x_2 are solutions, then $x_1 \equiv x_2 (mod\ I)$ and $x_1 \equiv x_2 (mod\ J)$. So $x_1 - x_2 \in I, J$, so $x_1 - x_2 \in I \cap J$. So $x_1 \equiv x_2 (mod\ I \cap J)$.*

(c) *Consider the map $f : R/(I \cap J) \to R/I \times R/J$ given by $f(x + I \cap J) = (x + I, x + J)$. If $x \equiv y (mod\ I \cap J)$, then $x - y \in I \cap J$, so $x - y \in I, J$ which gives $x \equiv y (mod\ I)$ and $x \equiv y (mod\ J)$. It is immediate to check it's a ring homomorphism. Part (b) tells us it's injective and part (a) tells us it's surjective.*

(6) Prove or disprove:

(a) The intersection of two ideals is always an ideal.

True. check the axioms routinely.

(b) The intersection of two prime ideals is always a prime ideal.

False. For example, inside $\mathbb{Z}$, $2\mathbb{Z}$ *and* $3\mathbb{Z}$ *are prime, but there intersection,* $6\mathbb{Z}$ *is not prime.*

(7) Do exercise 16.15 in Judson

If we look at the standard complex norm of things in $\mathbb{Z}\left[i\sqrt{5}\right]$*, no values between 0 and 1 are possible. So all units have norm 1. Now in* $\mathbb{Z}\left[i\sqrt{5}\right]$, $6 = 2 \cdot 3 = \left(1 - i\sqrt{5}\right)\left(1 + i\sqrt{5}\right)$*. The square of the norm of 2 is 4, of 3 is 9, and of* $1 \pm i\sqrt{5}$ *is 6. In* $\mathbb{Z}\left[i\sqrt{5}\right]$*, all numbers have square of norm an integer and 2 and 3 are impossible. So 2, 3, and* $1 \pm i\sqrt{5}$ *are all irreducible, and 2 and 3 don't divide* $1 \pm i\sqrt{5}$*. So the factorization of 6 is not unique.*