

Math 113 Homework 5 Solutions

- (1)  $Q_8$ , called the quaternion, is the set of size eight  $\{\pm 1, \pm i, \pm j, \pm k\}$  where  $(-1)x = x$ ,  $i^2 = j^2 = k^2 = -1$ , and

$$\begin{array}{lll} i \cdot j = k & j \cdot k = i & k \cdot i = j \\ j \cdot i = -k & k \cdot j = -i & i \cdot k = -j \end{array}$$

You do not need to verify  $Q_8$  is a group. (A presentation of  $Q_8$  in terms of matrices is given on page 43.)

Do exercise 9.3 in Judson.

*If a subgroup has some element other than 1, it must have  $-1$ . If a subgroup has anything besides 1 and  $-1$ , it has at least one of  $i$ ,  $j$ , or  $k$ . If it has two out of the three of  $i$ ,  $j$ , and  $k$ , or their negatives, then it is all of  $Q_8$ . So the subgroups are  $\{1\}$ ,  $\langle -1 \rangle$ ,  $\langle i \rangle$ ,  $\langle j \rangle$ ,  $\langle k \rangle$ , and  $Q_8$ . Since  $\langle i \rangle$ ,  $\langle j \rangle$ , and  $\langle k \rangle$  are index 2, they are normal. Also  $\langle -1 \rangle$  is the center, so all subgroups are normal.  $Q_8/\{1\} \cong Q_8$ , and  $Q_8/Q_8$  is trivial. Since they're each size 2,  $Q_8/\langle i \rangle$ ,  $Q_8/\langle j \rangle$ , and  $Q_8/\langle k \rangle$  are all isomorphic to  $\mathbb{Z}_2$ . Finally, note that for every  $q \in Q_8$ ,  $q^2 = \pm 1$ . So  $Q_8/\langle -1 \rangle$  every element has order 2, and  $Q_8/\langle -1 \rangle \not\cong \mathbb{Z}_4$ . By the previous homework,  $Q_8/\langle -1 \rangle$  must therefore be isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$ .*

- (2) If  $n$  divides  $m$ , Find an injective homomorphism  $f$  from  $\mathbb{Z}_n$  to  $\mathbb{Z}_m$ . Prove that  $\mathbb{Z}_m/\text{im}(f) \cong \mathbb{Z}_{m/n}$ .

Take  $f([k]_n) = \left[\frac{km}{n}\right]_m$ . First we need to check that  $f$  is well defined. If  $[k_1]_n = [k_2]_n$ , then for some integer  $\ell$ ,  $k_1 = k_2 + \ell n$ . Multiplying by  $\frac{m}{n}$ , we get  $\frac{k_1 m}{n} = \frac{k_2 m}{n} + \ell m$ . So  $\left[\frac{k_1 m}{n}\right]_m = \left[\frac{k_2 m}{n}\right]_m$ , i.e.  $f$  is well defined.

Next we verify that  $f$  is a homomorphism:

$$f([a]_n) + f([b]_n) = \left[\frac{am}{n}\right]_m + \left[\frac{bm}{n}\right]_m = \left[\frac{am}{n} + \frac{bm}{n}\right]_m = \left[\frac{(a+b)m}{n}\right]_m = f([ab]_n) = f([a]_n + [b]_n)$$

And then we verify that  $f$  is injective. If  $f([x]_n) = f([y]_n)$ , then  $\left[\frac{xm}{n}\right]_m = \left[\frac{ym}{n}\right]_m$ . So  $\frac{xm}{n} = \frac{ym}{n} + \ell m$  for some integer  $\ell$ . Multiplying by  $\frac{n}{m}$  gives  $x = y + \ell n$ . I.e.  $[x]_n = [y]_n$  and  $f$  is injective.

Now, since  $f$  is injective,  $\text{im}(f)$  has the same size as  $\mathbb{Z}_n$ ,  $n$ . So  $\mathbb{Z}_m/\text{im}(f)$  has size  $\frac{m}{n}$  by Lagrange's Theorem. Also  $\mathbb{Z}_m/\text{im}(f)$  is generated by  $[1]_m + \text{im}(f)$ . Since  $\mathbb{Z}_m/\text{im}(f)$  is a cyclic group of size  $\frac{m}{n}$ , it must be isomorphic to  $\mathbb{Z}_{m/n}$ .

- (3) Do exercise 9.18 in Judson.

*This is false. Take  $G = S_3$  and  $H = A_3$ . Then  $A_3$  has size 3, and  $S_3/A_3$  has size 2. Since 2 and 3 are prime,  $S_3$  and  $S_3/A_3$  must be abelian. Yet  $S_3$  is not abelian.*

- (4) Find a surjective homomorphism from  $A_4$  to  $\mathbb{Z}_3$ .

Since  $A_4$  is size 12 and  $\mathbb{Z}_3$  is size 3, we will need a kernel of size 4.  $A_4$  has only one subgroup of size 4,  $V = \{id, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$ . We showed in class that  $V$  is normal. (Alternatively, problem 6 tells us it is normal.) So  $A_4/V$  has size 3, and is therefore cyclic. i.e. for some  $\sigma \in A_4$ ,  $A_4/V = \langle \sigma V \rangle = \{V, gV, g^2V\}$ . So now we can map:

$$f(\nu) = \begin{cases} [0] & \nu \in V \\ [1] & \nu \in gV \\ [2] & \nu \in g^2V \end{cases}$$

Such an  $f$  is a surjective homomorphism from  $A_4$  to  $\mathbb{Z}_3$ .

(5) Do exercise 9.20 in Judson.

*This is false. Take  $G = \mathbb{Z}_2 \times \mathbb{Z}_2$  and  $H = \mathbb{Z}_2 \times \{[0]\}$ . Then  $H$  has size 2, and  $G/H$  has size 2. Since 2 is prime,  $H$  and  $G/H$  must be cyclic. Yet  $G$  is not cyclic.*

(6) Do exercise 9.23 in Judson.

*If  $H$  is a subgroup of size  $k$ , then for any  $g \in G$ , so is  $gHg^{-1}$ . Why is  $gHg^{-1}$  a subgroup? First  $e = geg^{-1} \in gHg^{-1}$ . Second, if  $gh_1g^{-1}, gh_2g^{-1} \in gHg^{-1}$ , then  $gh_1g^{-1}gh_2g^{-1} = gh_1h_2g^{-1} \in gHg^{-1}$ . Finally, if  $ghg^{-1} \in gHg^{-1}$ , then  $(ghg^{-1}) = (g^{-1})^{-1}h^{-1}g^{-1} = gh^{-1}g^{-1} \in gHg^{-1}$ . Also  $gHg^{-1}$  has the same size as  $H$ , because the map from  $H$  to  $gHg^{-1}$  given by  $x \rightarrow gxg^{-1}$  has the inverse  $y \rightarrow g^{-1}yg$ . So  $H$  and  $gHg^{-1}$  are in bijection.*

*Now, since  $G$  has only one subgroup of size  $k$ , it must be that  $H = gHg^{-1}$ . So  $H$  is normal.*

(7) Do exercise 9.26 parts (c) and (d) in Judson.

(c) *If  $g \in G$  and  $z \in Z(G)$ , then  $gz = zg$ . So  $gzg^{-1} = z$ . So  $gZ(g)g^{-1} \subset Z(G)$  for any  $g \in G$ . I.e.  $Z(G)$  is normal in  $G$ .*

(d) *If  $G/Z(G)$  is cyclic, then  $G/Z(G) = \langle gZ(g) \rangle$  for some  $g \in G$ . So for any  $x \in G$ , for some  $i \in \mathbb{Z}$ ,  $xZ(g) = (gZ(g))^i = g^iZ(G)$ . Thus  $x = g^iz$  for some  $i \in \mathbb{Z}$  and  $z \in Z(G)$ .*

*Now for any  $a, b \in G$ , we can write  $a = g^iz$  and  $b = g^jz'$  for some  $i, j \in \mathbb{Z}$  and  $z, z' \in Z(G)$ . Using the fact that  $z$  and  $z'$  commute with everything, we get:*

$$ab = g^izg^jz' = g^ig^jzz' = g^jg^iz'z = g^jz'g^iz = ba$$

*Thus  $G$  is commutative.*

(8) Do exercise 9.30 in Judson.

*We will give an injective homomorphism  $f : G \rightarrow G/H \times G/K$ . Since  $f$  is injective,  $\ker(f)$  will be trivial. So  $G \cong G/\ker(f) \cong \text{im}(f) \leq G/H \times G/K$ , which is what we want. Put  $f(g) = (gH, gK)$ . Then  $f(ab) = (abH, abK) = (aH, aK)(bH, bK) = f(a)f(b)$ , so  $f$  is a homomorphism.*

*To see that  $f$  is injective, assume  $f(a) = f(b)$ . So  $(aH, aK) = (bH, bK)$  or  $aH = bH$  and  $aK = bK$ . This tells us  $b^{-1}a \in H$  and  $b^{-1}a \in K$ . But  $H \cap K = \{e\}$ , so  $b^{-1}a = e$  or  $a = b$ . So  $f$  is injective, completing the proof.*