Math 113 Homework 6
due Monday July 27, 2009

(1) Let G be a group, N be a normal subgroup of G, and g be an element of G. Prove that the order of gN (in G/N) divides the order of g (in G).
(2) If $f : G \to H$ is a homomorphism, find a group K, and homomorphisms $g : G \to K$ and $h : K \to H$ so that g is surjective, h is injective, and $h \circ g = f$. Further, prove that K is unique up to isomorphism. (I.e. if the same is true for some other K', then $K \cong K'$.)
(3) Consider the ways of color the corners of a cube either red, blue, or green. Two colorings are the same if there is a rigid motion from one to the other. Use Burnside's Theorem to count the number of such colorings.
(4) Do exercise 12.20 in Judson.
(5) Do exercise 12.24 in Judson.
(6) Let G be a finite group, and p be a prime so that p divides the order of G. Define X to be the p-tuples from G whose product is the identity. I.e.

$$X = \{(g_1, g_2, \ldots, g_{p-1}, g_p) : g_1, g_2, \ldots, g_{p-1}, g_p \in G \text{ and } g_1 g_2 \ldots g_{p-1} g_p = e\}$$

Let σ be $(1\ 2\ 3\ \ldots\ p-1\ p)$ in S_p.
   (a) Prove that $|X| = |G|^{p-1}$.
   (b) We will let $\langle\sigma\rangle \leq S_p$ act on X by setting

$$\tau \star (g_1, g_2, \ldots, g_{p-1}, g_p) = \left(g_{\tau(1)}, g_{\tau(2)}, \ldots, g_{\tau(p-1)}, g_{\tau(p)}\right)$$

   First, prove by induction on i that if $(g_1, g_2, \ldots, g_{p-1}, g_p) \in X$ then $\sigma^i \star (g_1, g_2, \ldots, g_{p-1}, g_p) \in X$. Then prove that $\star$ is a valid group action.
   (c) Prove that p divides $|X_\sigma|$. (Use part (b) and the previous problem.)
   (d) Use (c) to show that G must have an element of order p.