

- (1) Let F be a commutative ring. Prove that F is a field if and only if F has exactly two ideals, $\{0\}$ and F .

Forwards: Suppose F is a field. Then $0 \neq 1$, so $\{0\} \neq F$ and F has at least two ideals. Suppose $I \neq \{0\}$ is an ideal of F , we will prove it equals F . Let i be a nonzero element of I and let a be any element of F . Then $(a\frac{1}{i})i = a$, so $a \in I$. Thus $I = F$.

Backwards: Suppose that F is a commutative ring with exactly two ideals, $\{0\}$ and F . Then $\{0\}$ is a maximal ideal. So $F/\{0\}$ is a field. But $F/\{0\}$ is isomorphic to F by the map that sends $x \rightarrow x + \{0\} = \{x\}$, so F is a field.

- (2) If f is a ring homomorphism from R to S , and I is an ideal of S , Show that $f^{-1}(I) = \{r \in R : f(r) \in I\}$ is an ideal of R .

Let g be the ring homomorphism from S to S/I given by $g(s) = s + I$. We know the kernel of g is I . Now we compute the kernel of $g \circ f$:

$$x \in \ker(g \circ f) \iff g(f(x)) = 0 \iff f(x) \in \ker(g) \iff f(x) \in I \iff x \in f^{-1}(I)$$

Since $f^{-1}(I) = \ker(f)$, it is an ideal.

- (3) Let S_3 act on the set of pairs $\{(i, j) : 1 \leq i, j \leq 3\}$ by $\sigma(i, j) = (\sigma(i), \sigma(j))$. Compute the orbits of this action. For each orbit, pick an element of the orbit and compute its stabilizer.

There are two orbits, one where both coordinates are the same, one where they are different. $\sigma(i, i) = (\sigma(i), \sigma(i))$. And if you want to send (i, i) to (j, j) , just pick a σ which sends i to j . If $i \neq j$, then $\sigma(i) \neq \sigma(j)$. Also, if you want to send (i, j) to (k, l) , where $i \neq j$ and $k \neq l$, just pick a σ so that $\sigma(i) = k$ and $\sigma(j) = l$. So the orbits are $\mathcal{O}_1 = \{(1, 1), (2, 2), (3, 3)\}$ and $\mathcal{O}_2 = \{(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)\}$.

Now for stabilizers: Since \mathcal{O}_2 has size 6, for any $x \in \mathcal{O}_2$, we know G_x has index 6. Since $|S_3|$ is 6, by Lagrange's theorem, it must be that $|G_x| = 1$, so it's just the identity. For example $G_{(1,2)} = \{id\}$. Similarly, if $x \in \mathcal{O}_1$, since $|\mathcal{O}_1| = 3$, G_x will have size 2. For a specific example, $G_{(1,1)} = \{id, (2\ 3)\}$.

- (4) Find a nontrivial group homomorphism from \mathbb{Z}_4 to \mathbb{Z}_6 . (Nontrivial means not always the identity.) Compute the image and kernel of your homomorphism.

Let $f([n]_4) = [3n]_6$. If $n \equiv m \pmod{4}$, then $n = m + 4k$ so $3n = 3m + 12k$ and $[3n]_6 = [3m]_6$. So f is well define. Also $f([i+j]_4) = [3(i+j)]_6 = [3i]_6 + [3j]_6 = f([i]_4) + f([j]_4)$, so f is a homomorphism. Finally, $f([1]_4) = [3]_6 \neq [0]_6$, so f is nontrivial.

The image of f is all things in \mathbb{Z}_6 of the form $[3n]_6$. That is just $\text{im}(f) = \{[0]_6, [3]_6\}$. The kernel of f is the $[n]_4$ so $[3n]_6 = [0]_6$. That happens when n is even. So $\ker(f) = \{[0]_4, [2]_4\}$.

- (5) Use Burnside's theorem to compute the number of ways to color the vertices of a triangle if n colors are used. Two colorings are considered the same if one is a reflection or rotation of the other.

The vertices of the triangle are being acted on by $D_3 = S_3$. X_{id} consists of all colorings, so it is size n^3 . For a transposition $(i j)$, i and j must be the same color, and the third vertex can be any color. So $|X_{(i j)}| = n^2$. Finally, for a 3-cycle, $(i j k)$, all three colors must be the same, so $|X_{(i j k)}| = n$. So by Burnside's theorem, the number of orbits, which is the number of colorings, is

$$\frac{1}{|S_3|} \sum_{\sigma \in S_3} |X_\sigma| = \frac{1}{6} (n^3 + 3n^2 + 2n)$$

- (6) Let G be a group, and let N and M be normal subgroups of G . Prove that $NM = \{nm : n \in N, m \in M\}$ is a normal subgroup of G .

The second isomorphism theorem tells us that NM is a group. To prove that NM is normal, let g be in G , and let nm be in NM so that $n \in N$ and $m \in M$. Then $g(nm)g^{-1} = gnmg^{-1} = gng^{-1}gmg^{-1}$. Since N and M are normal, $gng^{-1} \in N$ and $gmg^{-1} \in M$. Thus $g(nm)g^{-1} \in NM$. So for any $g \in G$, $gNMg^{-1} \subset NM$, and NM is normal.

- (7) If G acts on X , and for some $x, y \in X$ and $g \in G$, $gx = y$, prove that $G_y = gG_xg^{-1}$.

$G_y \subset gG_xg^{-1}$: Let a be in G_y , so $ay = y$. Now $g^{-1}agx = g^{-1}ay = g^{-1}y = g^{-1}gx = x$, so $g^{-1}ag \in G_x$ and $gg^{-1}agg^{-1} = a \in gG_xg^{-1}$.

$gG_xg^{-1} \subset G_y$: Let $b \in G_x$, so $bx = x$. Then $gbg^{-1}y = gbg^{-1}gx = gbx = gx = y$, so $gbg^{-1} \in G_y$ as desired.