

- (1) If G is a group, and H_1 and H_2 are subgroups of G , prove that $H_1 \cap H_2$ is a subgroup of G .

We will verify that $H_1 \cap H_2$ satisfies the conditions of the subgroup test of Proposition 2.9. First, since H_1 and H_2 are subgroups of G , e_G is in H_1 and H_2 . So e_G is in $H_1 \cap H_2$.

Next, if a and b are in $H_1 \cap H_2$, then a and b are both in both H_1 and H_2 . Since H_1 and H_2 are subgroups, ab is in H_1 and H_2 . So ab is in $H_1 \cap H_2$.

Lastly, if a is in $H_1 \cap H_2$, then a is in H_1 and H_2 . Since H_1 and H_2 are subgroups, a^{-1} is in both H_1 and H_2 . So a^{-1} is in $H_1 \cap H_2$.

Since $H_1 \cap H_2$ satisfies all the requirements of Proposition 2.9, it is a subgroup of G .

- (2) If G is a group, and H is a subgroup of G of index 2, prove that for any g in G , $gH = Hg$.

If g is in H , then $gH = H = Hg$ and we are done. So consider the case that g is not in H . The left cosets of H form a partition of G . Since H is one of the left cosets of H , and H has index two, this partition is just H and $G \setminus H$. Since g is not in H , $gH \neq H$. So it must be that $gH = G \setminus H$. By the same argument, the right cosets of H are just H and $G \setminus H$. Since g is not in H , $Hg \neq H$ so $Hg = G \setminus H = gH$. So in both cases, $gH = Hg$. (This is the same proof as in Corollary 5.10.)

- (3) Find a subgroup of A_4 of size four. Be sure to justify that it is a subgroup.

Such subgroup cannot have any 3-cycles. This is because 3-cycles have order three, and three does not divide four. There are only four elements of A_4 which are not 3-cycles: the identity and the products of pairs of disjoint transpositions. I.e.

$$V = \{id, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$$

By the way they are defined, each element of V is even, so V is a subset of A_4 . V has the identity, and each element of V is its own inverse. It remains to check that V is closed under composition. For any v in V , $v \circ v = id$ and $v \circ id = v = id \circ v$. To cover the remaining cases, note that for any distinct i, j, k, l , we get: $((i\ j)(k\ l)) \circ ((i\ k)(j\ l)) = (i\ l)(j\ k)$. I.e. the product of any two nonidentity elements of V is the other nonidentity element. So V satisfies Proposition 2.9 and is a subgroup of A_4 .

- (4) Recall that $H = \{id, (1\ 2)\}$ is a subgroup of S_3 . Explicitly compute all the left cosets and right cosets of H inside of S_3 .

The three left cosets of H are:

$$\begin{aligned} idH &= (1\ 2)H = H = \{id, (1\ 2)\} \\ (1\ 3)H &= (1\ 2\ 3)H = \{(1\ 3), (1\ 2\ 3)\} \\ (2\ 3)H &= (1\ 3\ 2)H = \{(2\ 3), (1\ 3\ 2)\} \end{aligned}$$

The three right cosets of H are:

$$\begin{aligned} Hid &= H(1\ 2) = H = \{id, (1\ 2)\} \\ H(1\ 3) &= H(1\ 3\ 2) = \{(1\ 3), (1\ 3\ 2)\} \\ H(2\ 3) &= H(1\ 2\ 3) = \{(2\ 3), (1\ 2\ 3)\} \end{aligned}$$

- (5) Prove that for any groups G and H , $G \times H$ is isomorphic to $H \times G$.

We will prove that the map $f : G \times H \rightarrow H \times G$ given by $f((g, h)) = (h, g)$ is an isomorphism. First, we verify that f is a homomorphism since for any (g_1, h_1) and (g_2, h_2) in $G \times H$ we get:

$$f((g_1, h_1)(g_2, h_2)) = f((g_1g_2, h_1h_2)) = (h_1h_2, g_1g_2) = (h_1, g_1)(h_2, g_2) = f((g_1, h_1))f((g_2, h_2))$$

To check that f is injective, suppose that $f((g_1, h_1)) = f((g_2, h_2))$. Then $(h_1, g_1) = (h_2, g_2)$. Which means $h_1 = h_2$ and $g_1 = g_2$. So $(g_1, h_1) = (g_2, h_2)$ and f is injective.

To check surjectivity, we just notice that for any (h, g) in $H \times G$, $f((g, h)) = (h, g)$. So $G \times H$ is isomorphic to $H \times G$ by way of f .

- (6) Find elements of S_7 of orders 10 and 12. Justify your answer.

We know that the order of a product of disjoint cycles is just the least common multiple of the lengths of the cycles. So if we take the product of a disjoint 2-cycle and 5-cycle, we'll get an element of order 10. For example $(1\ 2)(3\ 4\ 5\ 6\ 7)$ has order 10. Similarly, the product of a disjoint 3-cycle and 4-cycle will have order 12. One example in S_7 is $(1\ 2\ 3)(4\ 5\ 6\ 7)$.

- (7) Prove that if G is a group and every element of G has order at most two, then G must be abelian.

If every element of G has order at most two, then for each g in G , $gg = e_G$. Since inverses are unique, this means $g^{-1} = g$. Now for any a and b in G , we know $ab = (ab)^{-1} = b^{-1}a^{-1} = ba$. So G is abelian.