Math 1B Discussion Section Problems

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You should work on the following problems in groups of 3 or 4. Try to get through as many as you can, but you aren't expected to finish everything. Instead, you should make sure everyone in your group knows **how** to solve all the problems, and not just the answers.

Power Series Solutions

1. One of the hardest parts about using power series to solve these differential equations is in finding the pattern in the coefficients. For each of the following, "unwind" the recurrence relation:
   (a) $(n+1)a_{n+1} = a_n$ Write a_n in terms of a_0
   (b) $c_{n+2} = -\frac{c_n}{(n+1)(n+2)}$ Write c_n in terms of c_0 or c_1
   (c) $n^2 a_n + a_{n-2} = 0$ and $a_1 = 0$ Write a_n in terms of a_0
2. Find a power series solution to $y'' + x^2 y' + xy = 0$; $y(0) = 0, y'(0) = 1$
3. Use power series to find the general solution to $(x-1)y'' + y' = 0$. How else could you have solved this differential equation?
4. Use power series to solve $y'' + y = 0$. Is the answer what you expect?

Extra Problems If you finish early, take a stab at these.

1. Proving that e is irrational:
   (a) We'll do this by contradiction, so start by assuming that $e = \frac{p}{q}$ where p and q are both positive integers and $q > 2$. Use Taylor's remainder formula to explain why we can write
   $$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!} + \frac{e^z}{(q+1)!}$$
   where $0 < z < 1$
   (b) Show that since we are assuming $e = \frac{p}{q}$, then $q!(e - (1 + \frac{1}{1!} + \cdots + \frac{1}{q!}))$ must be an integer.
   (c) Show that $0 < q!(e - (1 + \frac{1}{1!} + \cdots + \frac{1}{q!})) < 1$
   (d) Explain why (b) and (c) show that e must not be writable in the form $\frac{p}{q}$
2. As you probably know, there are infinitely many prime numbers. Let's prove it:
   (a) Consider the series you would get by multiplying out $(1 + \frac{1}{2} + \frac{1}{4} + \cdots)(1 + \frac{1}{3} + \frac{1}{9} + \cdots)$. In terms of their prime factorization, what numbers would appear as denominators?
   (b) Do the same for $(1 + \frac{1}{2} + \frac{1}{4} + \cdots)(1 + \frac{1}{3} + \frac{1}{9} + \cdots)(1 + \frac{1}{5} + \frac{1}{25} + \cdots)$
   (c) By extrapolating from (a) and (b), what's another way of writing the product you get from using all the primes? ie, $(1 + \frac{1}{2} + \frac{1}{4} + \cdots)(1 + \frac{1}{3} + \frac{1}{9} + \cdots)(1 + \frac{1}{5} + \frac{1}{25} + \cdots)(1 + \frac{1}{7} + \frac{1}{49} + \cdots)(1 + \frac{1}{11} + \frac{1}{121} + \cdots)\cdots$ Does this converge or diverge?
   (d) Using the fact that each multiplicand has finite value (what is it?), show that there must be infinitely many prime numbers.
3. (A closed form for the Fibonacci sequence/One of Rob's favorite math facts) The Fibonacci sequence is defined by $F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}$
   (a) Use induction to show that if x satisfies the equation $x^2 = x + 1$, then $x^n = xF_n + F_{n-1}$ for any $n \geq 2$.
   Hint: $x^{n+1} = xx^n$
   (b) Let $y = \frac{-1+\sqrt{5}}{2}, z = \frac{-1-\sqrt{5}}{2}$ be the two roots of $x^2 = x + 1$. From part (a), we know that $y^n = yF_n + F_{n-1}$ and that $z^n = zF_n + F_{n-1}$. Subtract these equations and plug in the values of y and z to find a closed form for F_n.
   (c) Is it even obvious that your closed form evaluates to an integer?