Math 1B Discussion Section Problems

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You should work on the following problems in groups of 3 or 4. Try to get through as many as you can, but you aren't expected to finish everything. Instead, you should make sure everyone in your group knows **how** to solve all the problems, and not just the answers.

Some useful Taylor Series:

$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!};\ e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!};\ \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n;\ \ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n};\ \tan^{-1} x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$$

Binomial Series/Leftover Taylor Series

1. Use the binomial series formula to find a power series for $\sqrt{1+x}$ centered at 0.

2. Use Taylor's Remainder formula to derive the following useful[1] inequalities:

   (a) $|\ln(1+x) - x| \le \frac{1}{2}x^2$ whenever $x \ge 0$

   (b) $|\sin x - x| \le \frac{1}{6}|x|^3$ for all x

3. Let $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$.

   (a) Sketch a graph of this function.

   (b) Find $f'(0)$

   (c) It can be shown (though it's rather tedious) that $f^{(n)}(0)$ exists and is 0 for all n. What does this mean for the Taylor series of f at 0? What is its radius and interval of convergence?

   (d) For what values of x does $f(x)$ equal this Taylor Series?

4. (a) Expand $\frac{x+x^2}{(1-x)^3}$ as a power series by whatever means you feel like. Hint: repeatedly taking derivatives of this and using the Taylor series formula is probably **not** the correct answer.

   (b) Use (a) to find $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

Applications

1. Find $\displaystyle\lim_{x\to 0} \frac{x - \tan^{-1} x}{x^3}$

2. Determine whether each of the following series converge or diverge:

   (a) $\displaystyle\sum \frac{\sin \frac{1}{n}}{\sqrt{n}}$

   (b) $\sum \ln(n^2+1) - \ln(n^2)$

   (c) $\displaystyle\sum \frac{\sin \frac{1}{n^2}}{\tan^{-1} \frac{1}{n}}$

3. Proving that e is irrational:

   (a) We'll do this by contradiction, so start by assuming that $e = \frac{p}{q}$ where p and q are both positive integers and $q > 2$. Use Taylor's remainder formula to explain why we can write

   $$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!} + \frac{e^z}{(q+1)!}$$

   where $0 < z < 1$

[1] These are mostly useful because they're the mathematical justification for things like $\sin\theta \approx \theta$ near 0

(b) Show that since we are assuming $e = \frac{p}{q}$, then $q!(e - (1 + \frac{1}{1!} + \cdots + \frac{1}{q!}))$ must be an integer.

(c) Show that $0 < q!(e - (1 + \frac{1}{1!} + \cdots + \frac{1}{q!})) < 1$

(d) Explain why (b) and (c) show that e must not be writable in the form $\frac{p}{q}$

4. (a) Use Taylor series to find $\int e^{x^2}$

(b) How could you use your answer to (a) to find a good approximation to $\int_0^1 e^{x^2}$?

Extra Problems If you finish early, take a stab at these.

1. As you probably know, there are infinitely many prime numbers. Let's prove it:

(a) Consider the series you would get by multiplying out $(1 + \frac{1}{2} + \frac{1}{4} + \cdots)(1 + \frac{1}{3} + \frac{1}{9} + \cdots)$. In terms of their prime factorization, what numbers would appear as denominators?

(b) Do the same for $(1 + \frac{1}{2} + \frac{1}{4} + \cdots)(1 + \frac{1}{3} + \frac{1}{9} + \cdots)(1 + \frac{1}{5} + \frac{1}{25} + \cdots)$

(c) By extrapolating from (a) and (b), what's another way of writing the product you get from using all the primes? ie, $(1 + \frac{1}{2} + \frac{1}{4} + \cdots)(1 + \frac{1}{3} + \frac{1}{9} + \cdots)(1 + \frac{1}{5} + \frac{1}{25} + \cdots)(1 + \frac{1}{7} + \frac{1}{49} + \cdots)(1 + \frac{1}{11} + \frac{1}{121} + \cdots) \cdots$ Does this converge or diverge?

(d) Using the fact that each multiplicand has finite value (what is it?), show that there must be infinitely many prime numbers.

2. (A closed form for the Fibonacci sequence/One of Rob's favorite math facts) The Fibonacci sequence is defined by $F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}$

(a) Use induction to show that if x satisfies the equation $x^2 = x + 1$, then $x^n = xF_n + F_{n-1}$ for any $n \geq 2$.
Hint: $x^{n+1} = xx^n$

(b) Let $y = \frac{-1+\sqrt{5}}{2}, z = \frac{-1-\sqrt{5}}{2}$ be the two roots of $x^2 = x + 1$. From part (a), we know that $y^n = yF_n + F_{n-1}$ and that $z^n = zF_n + F_{n-1}$. Subtract these equations and plug in the values of y and z to find a closed form for F_n.

(c) Is it even obvious that your closed form evaluates to an integer?