Math 1B Discussion Section Problems

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You should work on the following problems in groups of 3 or 4. Try to get through as many as you can, but you aren't expected to finish everything. Instead, you should make sure everyone in your group knows **how** to solve all the problems, and not just the answers.

Some useful Taylor Series:

$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!};\ e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!};\ \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

More Taylor Series

1. Prove that e^x is equal to its Taylor series for all real numbers x
2. (a) Find the Maclaurin series for $\ln(1+x)$ and write it down somewhere.
   (b) What is the radius and interval of convergence?
   (c) Prove that this series converges to $\ln(1+x)$
   (d) There are at least 2 ways to find the Taylor series for $\ln(1+x)$. Figure out what the other one is and try it. Do you get the same answer? Which one was easier?
3. Use Taylor's Remainder formula to derive the following useful[1] inequalities:
   (a) $|\ln(1+x) - x| \leq \frac{1}{2}x^2$ whenever $x \geq 0$
   (b) $|\sin x - x| \leq \frac{1}{6}|x|^3$ for all x
4. (If you didn't get to it Friday...)
   Let $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$.
   (a) Sketch a graph of this function.
   (b) Find $f'(0)$
   (c) It can be shown (though it's rather tedious) that $f^{(n)}(0)$ exists and is 0 for all n. What does this mean for the Taylor series of f at 0? What is its radius and interval of convergence?
   (d) For what values of x does $f(x)$ equal this Taylor Series?

Binomial Series

1. Find a power series for $\sqrt{1+x}$
2. (a) Expand $\frac{x+x^2}{(1-x)^3}$ as a power series by whatever means you feel like.
   (b) Use (a) to find $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

Extra Problems If you finish early, take a stab at these.

1. As you probably know, there are infinitely many prime numbers. Let's prove it:
   (a) Consider the series you would get by multiplying out $(1 + \frac{1}{2} + \frac{1}{4} + \cdots)(1 + \frac{1}{3} + \frac{1}{9} + \cdots)$. In terms of their prime factorization, what numbers would appear as denominators?
   (b) Do the same for $(1 + \frac{1}{2} + \frac{1}{4} + \cdots)(1 + \frac{1}{3} + \frac{1}{9} + \cdots)(1 + \frac{1}{5} + \frac{1}{25} + \cdots)$
   (c) By extrapolating from (a) and (b), what's another way of writing the product you get from using all the primes? ie, $(1 + \frac{1}{2} + \frac{1}{4} + \cdots)(1 + \frac{1}{3} + \frac{1}{9} + \cdots)(1 + \frac{1}{5} + \frac{1}{25} + \cdots)(1 + \frac{1}{7} + \frac{1}{49} + \cdots)(1 + \frac{1}{11} + \frac{1}{121} + \cdots)\cdots$ Does this converge or diverge?

[1] These are mostly useful because they're the mathematical justification for things like $\sin\theta \approx \theta$ near 0

(d) Using the fact that each multiplicand has finite value (what is it?), show that there must be infinitely many prime numbers.

2. (A closed form for the Fibonacci sequence/One of Rob's favorite math facts) The Fibonacci sequence is defined by $F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}$

   (a) Use induction to show that if x satisfies the equation $x^2 = x + 1$, then $x^n = xF_n + F_{n-1}$ for any $n \geq 2$.
   Hint: $x^{n+1} = xx^n$

   (b) Let $y = \frac{-1+\sqrt{5}}{2}, z = \frac{-1-\sqrt{5}}{2}$ be the two roots of $x^2 = x + 1$. From part (a), we know that $y^n = yF_n + F_{n-1}$ and that $z^n = zF_n + F_{n-1}$. Subtract these equations and plug in the values of y and z to find a closed form for F_n.

   (c) Is it even obvious that your closed form evaluates to an integer?