Math 1B Discussion Section Problems

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You should work on the following problems in groups of 3 or 4. Try to get through as many as you can, but you aren't expected to finish everything. Instead, you should make sure everyone in your group knows **how** to solve all the problems, and not just the answers.

Taylor Series

1. What is the difference between a Taylor Series and a Maclaurin Series?

2. Find a Taylor Series expansion for each of the following functions and find the associated radius of convergence.

   (a) $f(x) = \sin x$, centered at $x = \pi/2$

   (b) $\sin^2(x)$, centered at 0. Hint: $2 \sin x \cos x = \sin(2x)$

   (c) $\frac{1}{\sqrt{x}}$, centered at 4.

3. We already know that $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$. Using this, find a Maclaurin series for:

   (a) $\sin(x^3)$

   (b) $x \sin(x^3)$

   (c) Can you use similar techniques to find a Taylor Series for $\sin(\sqrt{x+1})$? How about for $\cos(\sqrt{x+1})$?

4. (All about 0!)

   (a) How can you find 5! if you already know 4!?

   (b) How about 2! if you know 1!?

   (c) Why do the above two examples give a good hint as to why we should say $0! = 1$

5. Let $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$.

   (a) Sketch a graph of this function.

   (b) Find $f'(0)$

   (c) It can be shown (though it's rather difficult) that $f^{(n)}(0)$ exists and is 0 for all n. What does this mean for the Taylor series of f at 0? What is its radius of convergence?

   (d) For what values of x does $f(x)$ equal this Taylor Series?

Extra Problems If you finish early, take a stab at these.

1. As you probably know, there are infinitely many prime numbers. Let's prove it:

   (a) Consider the series you would get by multiplying out $(1 + \frac{1}{2} + \frac{1}{4} + \cdots)(1 + \frac{1}{3} + \frac{1}{9} + \cdots)$. In terms of their prime factorization, what numbers would appear as denominators?

   (b) Do the same for $(1 + \frac{1}{2} + \frac{1}{4} + \cdots)(1 + \frac{1}{3} + \frac{1}{9} + \cdots)(1 + \frac{1}{5} + \frac{1}{25} + \cdots)$

   (c) By extrapolating from (a) and (b), what's another way of writing the product you get from using all the primes? ie, $(1 + \frac{1}{2} + \frac{1}{4} + \cdots)(1 + \frac{1}{3} + \frac{1}{9} + \cdots)(1 + \frac{1}{5} + \frac{1}{25} + \cdots)(1 + \frac{1}{7} + \frac{1}{49} + \cdots)(1 + \frac{1}{11} + \frac{1}{121} + \cdots) \cdots$ Does this converge or diverge?

   (d) Using the fact that each multiplicand has finite value (what is it?), show that there must be infinitely many prime numbers.

2. (A closed form for the Fibonacci sequence/One of Rob's favorite math facts) The Fibonacci sequence is defined by $F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}$

   (a) Use induction to show that if x satisfies the equation $x^2 = x + 1$, then $x^n = xF_n + F_{n-1}$ for any $n \geq 2$.
   Hint: $x^{n+1} = xx^n$

   (b) Let $y = \frac{-1+\sqrt{5}}{2}, z = \frac{-1-\sqrt{5}}{2}$ be the two roots of $x^2 = x + 1$. From part (a), we know that $y^n = yF_n + F_{n-1}$ and that $z^n = zF_n + F_{n-1}$. Subtract these equations and plug in the values of y and z to find a closed form for F_n.

   (c) Is it even obvious that your closed form evaluates to an integer?

3. Show that if $\sum a_n x^n$ has infinitely many coefficients that are non-zero integers, then the radius of convergence is at most 1. Note: this is a hard problem to prove formally–if you're getting stuck with the symbols, try to figure out intuitively why this must be true.

   Hint 1: power series are absolutely convergent on any closed interval contained in their interval of convergence.

   Hint 2: It may be helpful to prove that any sub-series of an absolutely convergent series also converges.