# Math 1B Discussion Section Problems

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You should work on the following problems in groups of 3 or 4. Try to get through as many as you can, but you aren't expected to finish everything. Instead, you should make sure everyone in your group knows **how** to solve all the problems, and not just the answers.

**Power Series**

1. Starting from the geometric series, find a power series representation for each of the following functions, and determine the radius of convergence. DO NOT do a Taylor expansion.
   (a) $\dfrac{1}{1+x^3}$
   (b) $\dfrac{1}{4+x^2}$
   (c) $\arctan(x)$
   (d) $\dfrac{\ln(1+x)}{x}$
   (e) $\dfrac{1}{(1-x)^2}$
2. What function is represented by each of the following power series?
   (a) $\displaystyle\sum_{n=0}^{\infty} x^n$
   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n$
   (c) $\displaystyle\sum_{n=1}^{\infty} nx^{2n-1}$ [Hint: what if it were $2nx^{2n-1}$ inside the sum?]
3. (More proof that infinity is very, very strange)
   (a) Find a power series for $\dfrac{x}{(1-x)^2}$
   (b) Use your answer to (a) to find $\displaystyle\sum_{n=0}^{\infty} n\left(\frac{1}{2}\right)^n$?
   (c) What is $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$? Does your answer to (b) seem correct based on this?
4. You saw in class that differentiating a power series term-by-term doesn't change the radius or interval of convergence, except possibly at the endpoints. Let's show that this is a property unique to power series:
   (a) Show that $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ is absolutely convergent for all $x$.
   (b) Show that the series obtained by differentiating term-by-term diverges whenever $x = 2\pi, 4\pi, 6\pi, \ldots$
   (c) Why does this not contradict the theorem about differentiating power series term-by-term?
5. When we say $f(x) = \sum_{n=0}^{\infty} a_n x^n$, what we really mean is that these two things give the same value no matter what you plug in for $x$ (as long as it's in the interval of convergence).
   (a) In particular, when we plug in $x = 0$, both sides should give the same result. Based on this, what must $a_0$ be?

(b) If all the values of the power series and the function are the same, then all derivatives should be the same too (to see this, think about the formal definition of derivative). Using this, how can you determine what $a_1$ must be? $a_2$?

**Extra Problems** If you finish early, take a stab at these.

1. As you probably know, there are infinitely many prime numbers. Let's prove it:

   (a) Consider the series you would get by multiplying out $(1 + \frac{1}{2} + \frac{1}{4} + \cdots)(1 + \frac{1}{3} + \frac{1}{9} + \cdots)$. In terms of their prime factorization, what numbers would appear as denominators?

   (b) Do the same for $(1 + \frac{1}{2} + \frac{1}{4} + \cdots)(1 + \frac{1}{3} + \frac{1}{9} + \cdots)(1 + \frac{1}{5} + \frac{1}{25} + \cdots)$

   (c) By extrapolating from (a) and (b), what's another way of writing the product you get from using all the primes? ie, $(1 + \frac{1}{2} + \frac{1}{4} + \cdots)(1 + \frac{1}{3} + \frac{1}{9} + \cdots)(1 + \frac{1}{5} + \frac{1}{25} + \cdots)(1 + \frac{1}{7} + \frac{1}{49} + \cdots)(1 + \frac{1}{11} + \frac{1}{121} + \cdots)\cdots$ Does this converge or diverge?

   (d) Using the fact that each multiplicand has finite value (what is it?), show that there must be infinitely many prime numbers.

2. (A closed form for the Fibonacci sequence) The Fibonacci sequence is defined by $F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}$

   (a) Use induction to show that if $x$ satisfies the equation $x^2 = x + 1$, then $x^n = xF_n + F_{n-1}$ for any $n \geq 2$.
   Hint: $x^{n+1} = xx^n$

   (b) Let $y = \frac{-1+\sqrt{5}}{2}, z = \frac{-1-\sqrt{5}}{2}$ be the two roots of $x^2 = x + 1$. From part (a), we know that $y^n = yF_n + F_{n-1}$ and that $z^n = zF_n + F_{n-1}$. Subtract these equations and plug in the values of $y$ and $z$ to find a closed form for $F_n$.

   (c) Is it even obvious that your closed form evaluates to an integer?

3. Show that if $\sum a_n x^n$ has infinitely many coefficients that are non-zero integers, then the radius of convergence is at most 1. Note: this is a hard problem to prove formally–if you're getting stuck with the symbols, try to figure out intuitively why this must be true.

   Hint 1: power series are absolutely convergent on any closed interval contained in their interval of convergence.

   Hint 2: It may be helpful to prove that any sub-series of an absolutely convergent series also converges.