Math 1B Discussion Section Problems

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You should work on the following problems in groups of 3 or 4. Try to get through as many as you can, but you aren't expected to finish everything. Instead, you should make sure everyone in your group knows **how** to solve all the problems, and not just the answers.

Series Mish-Mash

1. Use whichever test you feel like to determine whether each of the following series is absolutely convergent, conditionally convergent, or divergent. Are there any other tests that would have worked? You may start the series anywhere you feel is convenient.

   (a) $\sum \frac{\tan^{-1} n}{n^3}$

   (b) $\sum \frac{n!}{e^{n^2}}$

   (c) $\sum \frac{n!}{2 \cdot 5 \cdot 8 \cdots (3n+2)}$

   (d) $\sum \frac{(-1)^n}{n\sqrt{\ln n}}$

   (e) $\sum \frac{1}{1+e^n}$

   (f) $\sum \left(\frac{1}{n^{5/7}} - \frac{1}{(n+1)^{5/7}} \right)$

   (g) $\sum \frac{\cos(\pi n)}{n(\ln n)^2}$

   (h) $\sum (-1)^n \frac{3^n}{n^3}$

   (i) $\sum \frac{(-2)^{3n}}{n^n}$

   (j) $\sum \ln \frac{n+2}{n}$

   (k) $\sum \frac{\sqrt{n+1}-\sqrt{n}}{n}$

   (l) $\sum (-1)^n \sqrt[n]{2}$

2. Determine whether $\sum \frac{\sin(1/n)}{\sqrt{n}}$ is AC, CC, or D. How about $\sum \frac{\sin \frac{1}{n^3}}{\sin \frac{1}{n}}$? Hint: what do you know about $\sin\theta$ for $\theta \approx 0$?

3. For what values of p is $\sum \frac{(-1)^n}{n^p}$ AC? CC? D?

4. For what values of p does $\sum \frac{\ln n}{n^p}$ converge?

5. Determine whether $\sum \frac{(-1)^n}{n+(-1)^n}$ converges or diverges. What about $\sum \frac{(-1)^n}{n^2+(-1)^n}$?

Extra Problems If you finish early, take a stab at these.

1. As you probably know, there are infinitely many prime numbers. Let's prove it:

   (a) Consider the series you would get by multiplying out $(1+\frac{1}{2}+\frac{1}{4}+\cdots)(1+\frac{1}{3}+\frac{1}{9}+\cdots)$. In terms of their prime factorization, what numbers would appear as denominators?

(b) Do the same for $(1+\frac{1}{2}+\frac{1}{4}+\cdots)(1+\frac{1}{3}+\frac{1}{9}+\cdots)(1+\frac{1}{5}+\frac{1}{25}+\cdots)$

(c) By extrapolating from (a) and (b), what's another way of writing the product you get from using all the primes? ie, $(1+\frac{1}{2}+\frac{1}{4}+\cdots)(1+\frac{1}{3}+\frac{1}{9}+\cdots)(1+\frac{1}{5}+\frac{1}{25}+\cdots)(1+\frac{1}{7}+\frac{1}{49}+\cdots)(1+\frac{1}{11}+\frac{1}{121}+\cdots)\cdots$ Does this converge or diverge?

(d) Using the fact that each multiplicand has finite value (what is it?), show that there must be infinitely many prime numbers.

2. (A closed form for the Fibonacci sequence) The Fibonacci sequence is defined by $F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}$

   (a) Use induction to show that if x satisfies the equation $x^2 = x + 1$, then $x^n = xF_n + F_{n-1}$ for any $n \geq 2$.
   Hint: $x^{n+1} = xx^n$

   (b) Let $y = \frac{-1+\sqrt{5}}{2}, z = \frac{-1-\sqrt{5}}{2}$ be the two roots of $x^2 = x + 1$. From part (a), we know that $y^n = yF_n + F_{n-1}$ and that $z^n = zF_n + F_{n-1}$. Subtract these equations and plug in the values of y and z to find a closed form for F_n.

   (c) Is it even obvious that your closed form evaluates to an integer?

3. Find the flaw in the following "proof" that 0=1:

$$\begin{aligned} 0 &= 0+0+0+\cdots \\ &= (1-1)+(1-1)+(1-1)+\cdots \\ &= 1-1+1-1+1-1+\cdots \\ &= 1+(-1+1)+(-1+1)+\cdots \\ &= 1+0+0+0+\cdots \\ &= 1 \end{aligned}$$

4. True/false. For those that are true, provide a brief explanation/intuition of why. For those that are false, find a counterexample:

   (a) If a_n is positive for all n, and each partial sum is less than 10^4, then $\sum_{n=0}^{\infty} a_n$ converges

   (b) If $a_n < b_n$ for all n and both sequences converge, then $\lim a_n < \lim b_n$

   (c) If s_n is the sequence of partial sums for the sequence a_n and $\lim_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} s_n$ exists.

   (d) If s_n is the sequence of partial sums for $\sum a_n$ and $\lim_{n\to\infty} s_n = 0$, then $\lim_{n\to\infty} a_n = 0$