

# Math 1B Discussion Section Problems

Rob Bayer

August 4, 2008

You should work on the following problems in groups of 3 or 4. Try to get through as many as you can, but you aren't expected to finish everything. Instead, you should make sure everyone in your group knows **how** to solve all the problems, and not just the answers.

## Ratio Test

1. Determine whether each of the following series is absolutely convergent, conditionally convergent, or diverges:

(a)  $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

(b)  $\sum_{n=1}^{\infty} \frac{n3^n}{5^{n-1}}$

(c)  $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^n$

(d)  $\sum_{n=1}^{\infty} \frac{(n^4 + 1)2^{2n}}{5^{n+3}}$

(e)  $\sum_{n=1}^{\infty} \frac{n!}{n^n}$  Hint:  $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$

(f)  $\sum_{n=1}^{\infty} \frac{\sin(4n)}{n^2}$

(g)  $\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdots 2n}{(2n)!}$

2. We said that the ratio test gives no information when  $L = 1$ . Prove this by giving an example of a series with  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1$  and

- (a)  $\sum a_n$  is Absolutely Convergent
- (b)  $\sum a_n$  is conditionally convergent
- (c)  $\sum a_n$  is divergent

3. Let  $p(n)$  be a polynomial of degree  $l$  and  $q(n)$  be a polynomial of degree  $k$ . What happens when you apply the ratio test to  $\sum \frac{p(n)}{q(n)}$ ? What conditions on  $l$  and  $k$  make this converge? Diverge?

4. True/False. Prove or provide counterexamples.

- (a) If  $L = 1$  in the ratio test, then  $\sum a_n$  is conditionally convergent.
- (b) If  $\sum a_n$  is conditionally convergent, then  $\lim \left| \frac{a_{n+1}}{a_n} \right|$  either DNE or is 1.
- (c) If  $\sum a_n$  and  $\sum b_n$  are divergent series with only non-negative terms, then  $\sum (a_n + b_n)$  diverges.
- (d) If  $\sum a_n$  converges and  $a_n > 0$ , then  $\sum (-1)^n a_n$  converges

5. For which values of  $x$  does  $\sum_{n=0}^{\infty} nx^n$  converge?

6. For what values of  $r$  and  $p$  will  $\sum r^n n^p$  converge? What about  $\sum r^n (n!)^p$ ?

7. Prove that if  $\sum a_n$  converges and  $|x| < 1$ , then  $\sum a_n x^n$  converges absolutely.

**Extra Problems** If you finish early, take a stab at these.

1. As you probably know, there are infinitely many prime numbers. Let's prove it:
  - (a) Consider the series you would get by multiplying out  $(1 + \frac{1}{2} + \frac{1}{4} + \dots)(1 + \frac{1}{3} + \frac{1}{9} + \dots)$ . In terms of their prime factorization, what numbers would appear as denominators?
  - (b) Do the same for  $(1 + \frac{1}{2} + \frac{1}{4} + \dots)(1 + \frac{1}{3} + \frac{1}{9} + \dots)(1 + \frac{1}{5} + \frac{1}{25} + \dots)$
  - (c) By extrapolating from (a) and (b), what's another way of writing the product you get from using all the primes? ie,  $(1 + \frac{1}{2} + \frac{1}{4} + \dots)(1 + \frac{1}{3} + \frac{1}{9} + \dots)(1 + \frac{1}{5} + \frac{1}{25} + \dots)(1 + \frac{1}{7} + \frac{1}{49} + \dots)(1 + \frac{1}{11} + \frac{1}{121} + \dots) \dots$  Does this converge or diverge?
  - (d) Using the fact that each multiplicand has finite value (what is it?), show that there must be infinitely many prime numbers.
  
2. (A closed form for the Fibonacci sequence) The Fibonacci sequence is defined by  $F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}$ 
  - (a) Use induction to show that if  $x$  satisfies the equation  $x^2 = x + 1$ , then  $x^n = xF_n + F_{n-1}$  for any  $n \geq 2$ .  
Hint:  $x^{n+1} = xx^n$
  - (b) Let  $y = \frac{-1+\sqrt{5}}{2}, z = \frac{-1-\sqrt{5}}{2}$  be the two roots of  $x^2 = x + 1$ . From part (a), we know that  $y^n = yF_n + F_{n-1}$  and that  $z^n = zF_n + F_{n-1}$ . Subtract these equations and plug in the values of  $y$  and  $z$  to find a closed form for  $F_n$ .
  - (c) Is it even obvious that your closed form evaluates to an integer?