Math 1B Discussion Section Problems

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You should work on the following problems in groups of 3 or 4. Try to get through as many as you can, but you aren't expected to finish everything. Instead, you should make sure everyone in your group knows **how** to solve all the problems, and not just the answers.

Alternating Series

1. For each of the following, determine (a) whether it is an alternating series and (b) if it converges conditionally, converges absolutely, or diverges.

   (a) $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+1}$

   (b) $\sum_{n=1}^{\infty}(-1)^n \frac{\cos(\pi n)}{\sqrt{n}}$

   (c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{3^n}$

   (d) $\sum_{n=1}^{\infty}(-1)^n \cos\left(\frac{\pi}{n}\right)$

   (e) $\sum \frac{\sin n}{n^2}$

2. We've now seen how to deal with series with only positive terms, and series with alternating sign terms. How would you approach testing a series with only negative terms?

3. Show that even though $\lim b_n = 0$, the series $\sum(-1)^{n-1}b_n$, where $b_n = \begin{cases} 1/n & \text{if n is odd} \\ 1/n^2 & \text{if n is even} \end{cases}$, diverges.

4. Give an example of each of the following situations:

   (a) $\sum a_n$ converges, but $\sum a_n^2$ diverges

   (b) $\sum a_n$ diverges, but $\sum a_n^2$ converges

5. Prove or give a counterexample:

   (a) If $\sum a_n$ is absolutely convergent, then $\sum a_n^2$ converges.

   (b) If $\sum a_n$ and $\sum b_n$ are conditionally convergent, then $\sum a_n b_n$ is too.

Extra Problems If you finish early, take a stab at these.

1. (Infinitely many prime numbers)

   (a) Consider the series you would get by multiplying out $(1+\frac{1}{2}+\frac{1}{4}+\cdots)(1+\frac{1}{3}+\frac{1}{9}+\cdots)$. In terms of their prime factorization, what numbers would appear as denominators?

   (b) Do the same for $(1+\frac{1}{2}+\frac{1}{4}+\cdots)(1+\frac{1}{3}+\frac{1}{9}+\cdots)(1+\frac{1}{5}+\frac{1}{25}+\cdots)$

   (c) By extrapolating from (a) and (b), what's another way of writing the product you get from using all the primes? ie, $(1+\frac{1}{2}+\frac{1}{4}+\cdots)(1+\frac{1}{3}+\frac{1}{9}+\cdots)(1+\frac{1}{5}+\frac{1}{25}+\cdots)(1+\frac{1}{7}+\frac{1}{49}+\cdots)(1+\frac{1}{11}+\frac{1}{121}+\cdots)\cdots$ Does this converge or diverge?

   (d) Using the fact that each multiplicands has finite value (what is it?) to show that there must be infinitely many prime numbers.

2. (A closed form for the Fibonacci sequence) The Fibonacci sequence is defined by $F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}$

(a) Use induction to show that if x satisfies the equation $x^2 = x + 1$, then $x^n = xF_n + F_{n-1}$ for any $n \geq 2$.
Hint: $x^{n+1} = xx^n$

(b) Let $y = \frac{-1+\sqrt{5}}{2}$, $z = \frac{-1-\sqrt{5}}{2}$ be the two roots of $x^2 = x + 1$. From part (a), we know that $y^n = yF_n + F_{n-1}$ and that $z^n = zF_n + F_{n-1}$. Subtract these equations and plug in the values of y and z to find a closed form for F_n.

(c) Is it even obvious that your closed form evaluates to an integer?