Math 1B Discussion Section Problems

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July 31, 2008

You should work on the following problems in groups of 3 or 4. Try to get through as many as you can, but you aren't expected to finish everything. Instead, you should make sure everyone in your group knows **how** to solve all the problems, and not just the answers.

Comparison Test

1. Determine whether each of the following series converge or diverge.
   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n-3}{n^3+4n+2}$
   (b) $\displaystyle\sum_{n=2}^{\infty} \frac{2+\sin n}{n^2+\ln n}$
   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+e^n}$
   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n2^n}$
   (e) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$
2. (a) Show that if $\sum a_n$ is a convergent series with non-negative terms, then $\sum a_n^2$ is also convergent.
   (b) However, if $\sum a_n$ is a convergent series with non-negative terms, $\sum \sqrt{a_n}$ could either converge or diverge. Give an example for each of these possibilities.

Limit Comparison Test

1. Determine whether each of the following series converge or diverge.
   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{3n^3+4n}{2^n(n^3+6n^2)}$
   (b) $\displaystyle\sum_{n=10}^{\infty} \frac{2n+3}{\sqrt[3]{n^5+3n^4-6n^2+1}}$
2. Despite first appearances, the limit comparison test is not always better than the comparison test. What happens when you try to apply it to $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2}$?
3. Let p_n denote the nth prime number. For example, $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, p_5 = 11$, etc. A very important (and difficult!) theorem in number theory states that $\displaystyle\lim_{n\to\infty} \frac{p_n}{n \ln n} = 1$. Use this to show that $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots$ diverges to ∞.
4. (a) Suppose a_n and b_n are sequences of positive numbers. Using the definition of convergence of a sequence and the comparison test, prove that
      i. If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
      ii. If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.
   (b) Give an example of a pair of series $\sum a_n, \sum b_n$ with positive terms such that $\lim \frac{a_n}{b_n} = 0$ and $\sum b_n$ diverges but $\sum a_n$ converges.
   (c) Use part (a) to show that if $\sum a_n$ and $\sum b_n$ are both convergent series with positive terms, then $\sum a_n b_n$ is convergent too.

Extra Problems If you finish early, take a stab at these.

1. The Cantor Set is a set of real numbers constructed as follows: start with the interval $[0, 1]$, and remove the middle third of it. That is, remove the interval $(\frac{1}{3}, \frac{2}{3})$, leaving $[0, \frac{1}{3}], [\frac{2}{3}, 1]$. Now remove the middle third of each of these remaining intervals, leaving $[0, \frac{1}{9}], [\frac{2}{9}, \frac{1}{3}], [\frac{2}{3}, \frac{7}{9}], [\frac{8}{9}, 1]$. After continuing this process infinitely many times, you will be left with the Cantor set.

   (a) Show that the total length of all the intervals you remove is 1.

   (b) Convince yourselves that despite this, the cantor set has infinitely many numbers in it. Give some examples of these numbers.

   (c) (Side note: it actually turns out that the Cantor Set is uncountable, meaning there are exactly the same number of numbers in it as there were in the interval $[0, 1]$ before you started removing middle thirds. The proof of this is actually very easy, but requires some knowledge of binary and ternary decimal systems. Talk to me if you're curious.)

2. Let's find a closed form for the Fibonacci sequence, which is defined by $F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}$

   (a) Use induction to show that if x satisfies the equation $x^2 = x + 1$, then $x^n = xF_n + F_{n-1}$ for any $n \geq 2$.
   Hint: $x^{n+1} = xx^n$

   (b) Let $y = \frac{-1+\sqrt{5}}{2}, z = \frac{-1-\sqrt{5}}{2}$ be the two roots of $x^2 = x + 1$. From part (a), we know that $y^n = yF_n + F_{n-1}$ and that $z^n = zF_n + F_{n-1}$. Subtract these equations and plug in the values of y and z to find a closed form for F_n.