

# Math 1B Discussion Section Problems

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You should work on the following problems in groups of 3 or 4. Try to get through as many as you can, but you aren't expected to finish everything. Instead, you should make sure everyone in your group knows **how** to solve all the problems, and not just the answers.

## Telescoping and Other Series

1. Determine whether each of the following telescoping series are convergent or divergent. Evaluate the convergent ones

(a)  $\sum_{n=1}^{\infty} \sin(n+1) - \sin(n)$

(b)  $\sum_{n=2}^{\infty} \ln \frac{n^2 + 2n + 1}{n^2}$

(c)  $\sum_{n=1}^{\infty} \frac{-2}{n(n+2)}$

2. For what values of  $x$  does  $\sum_{n=0}^{\infty} \frac{\cos^n x}{2^n}$  converge? What is its value for those values of  $x$ ?

3. True/False. For those that are true, prove it. For those that are false, provide a counterexample.

(a) If  $\sum a_n$  converges, then  $\sum \frac{1}{a_n}$  diverges. Assume  $a_n \neq 0$  for any  $n$

(b) If  $\sum a_n$  diverges, then  $\sum (-1)^n a_n$  diverges.

(c) If  $\sum a_n$  diverges and  $\sum b_n$  diverges, then  $\sum a_n + b_n$  diverges.

(d) If  $\sum a_n$  converges and  $\sum b_n$  diverges, then  $\sum a_n + b_n$  diverges.

4. At some point, you may start wondering why we only deal with infinite sums, and not with infinite products, which are denoted by  $\prod_{n=1}^{\infty} a_n$ . As with series, we define infinite products as a limit:

$$\prod_{n=1}^{\infty} a_n := \lim_{n \rightarrow \infty} p_n \text{ where } p_N = \prod_{n=1}^N a_n$$

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(a) Show that  $\ln p_N = \sum_{n=1}^N \ln a_n$

(b) Use this to show that  $\prod_{n=1}^{\infty} a_n = e^{\sum \ln a_n}$  and conclude that in order to compute infinite products, it is enough to be able to compute infinite series.

(c) Show that if  $\prod_{n=1}^{\infty} a_n$  converges, then  $\lim a_n = 1$

5. Use the fact that  $\arctan(n+1) = \arctan(n) + \arctan \frac{1}{n^2+n+1}$  to compute  $\sum_{n=0}^{\infty} \arctan \frac{1}{n^2+n+1}$

6. Find  $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \frac{1}{n(n+1)}$

## Crazy Facts

1. Find the flaw in the following “proof” that  $0=1$ :

$$\begin{aligned}
 0 &= 0 + 0 + 0 + \dots \\
 &= (1 - 1) + (1 - 1) + (1 - 1) + \dots \\
 &= 1 - 1 + 1 - 1 + 1 - 1 + \dots \\
 &= 1 + (-1 + 1) + (-1 + 1) + \dots \\
 &= 1 + 0 + 0 + 0 + \dots \\
 &= 1
 \end{aligned}$$

2. The Cantor Set is a set of real numbers constructed as follows: start with the interval  $[0, 1]$ , and remove the middle third of it. That is, remove the interval  $(\frac{1}{3}, \frac{2}{3})$ , leaving  $[0, \frac{1}{3}]$ ,  $[\frac{2}{3}, 1]$ . Now remove the middle third of each of these remaining intervals, leaving  $[0, \frac{1}{9}]$ ,  $[\frac{2}{9}, \frac{1}{3}]$ ,  $[\frac{2}{3}, \frac{7}{9}]$ ,  $[\frac{8}{9}, 1]$ . After continuing this process infinitely many times, you will be left with the Cantor set.

- Show that the total length of all the intervals you remove is 1.
- Convince yourselves that despite this, the cantor set has infinitely many numbers in it. Give some examples of these numbers.
- (Side note: it actually turns out that the Cantor Set is uncountable, meaning there are exactly the same number of numbers in it as there were in the interval  $[0, 1]$  before you started removing middle thirds. The proof of this is actually very easy, but requires some knowledge of binary and ternary decimal systems. Talk to me if you're curious.)

3. Recall that the Fibonacci sequence is defined as  $F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}$

- If you haven't before, write out a few terms of this sequence to get a sense of the pattern.
- Show that  $\frac{1}{F_{n-1}F_{n+1}} = \frac{1}{F_{n-1}F_n} - \frac{1}{F_nF_{n+1}}$
- Show that  $\sum_{n=2}^{\infty} \frac{1}{F_{n-1}F_{n+1}} = 1$
- Show that  $\sum_{n=2}^{\infty} \frac{F_n}{F_{n-1}F_{n+1}} = 2$

### Extra Problems

1. True/false. For those that are true, provide a brief explanation/intuition of why. For those that are false, find a counterexample:

- If  $a_n$  is positive for all  $n$ , and each partial sum is less than  $10^4$ , then  $\sum_{n=0}^{\infty} a_n$  converges
- If  $a_n < b_n$  for all  $n$  and both sequences converge, then  $\lim a_n < \lim b_n$
- If  $s_n$  is the sequence of partial sums for the sequence  $a_n$  and  $\lim_{n \rightarrow \infty} a_n = 0$ , then  $\lim_{n \rightarrow \infty} s_n$  exists.
- If  $s_n$  is the sequence of partial sums for  $\sum a_n$  and  $\lim_{n \rightarrow \infty} s_n = 0$ , then  $\lim_{n \rightarrow \infty} a_n = 0$

2. Let's find a closed form for the Fibonacci sequence, which is defined by  $F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}$

- Use induction to show that if  $x$  satisfies the equation  $x^2 = x + 1$ , then  $x^n = xF_n + F_{n-1}$  for any  $n \geq 2$ .  
Hint:  $x^{n+1} = xx^n$
- Let  $y = \frac{-1+\sqrt{5}}{2}, z = \frac{-1-\sqrt{5}}{2}$  be the two roots of  $x^2 = x + 1$ . From part (a), we know that  $y^n = yF_n + F_{n-1}$  and that  $z^n = zF_n + F_{n-1}$ . Subtract these equations and plug in the values of  $y$  and  $z$  to find a closed form for  $F_n$ .

3. True/False. For all problems,  $a_n$  and  $b_n$  are sequences. Justify your answers with a sketch of a proof or a counterexample.

- If  $a_n$  and  $b_n$  converge, then  $a_n + b_n$  converges.
- If  $a_n + b_n$  converges, then  $a_n$  and  $b_n$  converge.
- If  $a_n$  and  $b_n$  converge, then  $a_n/b_n$  converges.
- If  $a_n$  and  $b_n$  diverge, then  $a_n + b_n$  diverges.
- If  $a_n + b_n$  diverges, then  $a_n$  and  $b_n$  diverge.
- If  $a_n$  and  $b_n$  diverge, then  $a_nb_n$  diverges.