Math 1B Discussion Section Problems

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You should work on the following problems in groups of 3 or 4. Try to get through as many as you can, but you aren't expected to finish everything. Instead, you should make sure everyone in your group knows **how** to solve all the problems, and not just the answers.

Series I

1. Re-write each of the following in $\sum$ notation. Start your sum wherever is convenient.
   (a) $1 + 2 + 3 + \cdots$
   (b) $1 + \frac{1}{4} + \frac{1}{9} + \cdots$
   (c) $1 - 1 + 1 - 1 + \cdots$
   (d) $\frac{3}{2} - \frac{3}{4} + \frac{3}{8} - \frac{3}{16} + \cdots$

2. Re-index each of the following series to start at $n = 0$
   (a) $\displaystyle\sum_{n=1}^{\infty} ar^{n-1}$
   (b) $\displaystyle\sum_{n=2}^{\infty} \ln\left(\frac{n+1}{n+2}\right)$
   (c) $\displaystyle\sum_{n=-1}^{\infty} \sin^n\left(\frac{n\pi}{2}\right)$

3. Determine whether each of the following series are convergent or divergent. For those that are convergent, find the sum.
   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n+2}}{2^{2n}}$
   (b) $\displaystyle\sum_{n=1}^{\infty} \sqrt[n]{2}$
   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n}$
   (d) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{2+3^n}{4^n}$
   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-3}{n^2+3n+1}$

4. If the nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ is $s_n = \frac{n-1}{n+1}$, find a_n and $\sum_{n=1}^{\infty} a_n$

5. (A Preview Of Power Series) For which values of x do each of the following converge?
   (a) $\sum_{n=0}^{\infty} x^n$
   (b) $\sum_{n=0}^{\infty} x^n 2^n$
   (c) $\sum_{n=1}^{\infty} \frac{x}{n}$
   (d) $\sum_{n=0}^{\infty} \frac{(x+1)^n}{3^{2n+1}}$

6. True/false. For those that are true, provide a brief explanation/intuition of why. For those that are false, find a counterexample:

(a) If a_n is positive for all n, and each partial sum is less than 10^4, then $\sum_{n=0}^{\infty} a_n$ converges

(b) If $a_n < b_n$ for all n and both sequences converge, then $\lim a_n < \lim b_n$

(c) If s_n is the sequence of partial sums for the sequence a_n and $\lim_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} s_n$ exists.

Applications

1. (Zeno's Paradox) Suppose you are 1 meter away from a wall and want to walk up and touch it. Then you must first go half the distance to the wall, which takes some positive amount of time, then half of the remaining distance, then half of what still remains, etc, etc. Show that despite this, it only takes a finite amount of time to walk across the room. Assume you can move at 1m/s.

2. Prove that $.\overline{9} = 1$

3. Re-write $3.\overline{417}$ as a rational number.

4. A certain ball has the property that each time it falls from height h onto a hard, level surface, it rebounds to a height $rh, (0 < r < 1)$. What is the total distance travelled by the ball if it is dropped from height H and bounces indefinitely?

Extra Problems

1. Let's find a closed form for the Fibonacci sequence, which is defined by $F_1 = 1, F_2 = 2, F_n = F_{n-1} + F_{n-2}$

   (a) Use induction to show that if x satisfies the equation $x^2 = x + 1$, then $x^n = xF_n + F_{n-1}$.
   Hint: $x^{n+1} = xx^n$

   (b) Let $y = \frac{-1+\sqrt{5}}{2}, z = \frac{-1-\sqrt{5}}{2}$ be the two roots of $x^2 = x + 1$. From part (a), we know that $y^n = yF_n + F_{n-1}$ and that $z^n = zF_n + F_{n-1}$. Subtract these equations and plug in the values of y and z to find a closed form for F_n.

2. True/False. For all problems, a_n and b_n are sequences. Justify your answers with a sketch of a proof or a counterexample.

   (a) If a_n and b_n converge, then $a_n + b_n$ converges.

   (b) If $a_n + b_n$ converges, then a_n and b_n converge.

   (c) If a_n and b_n converge, then a_n/b_n converges.

   (d) If a_n and b_n diverge, then $a_n + b_n$ diverges.

   (e) If $a_n + b_n$ diverges, then a_n and b_n diverge.

   (f) If a_n and b_n diverge, then a_nb_n diverges.