Math 1B Discussion Section Problems

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You should work on the following problems in groups of 3 or 4. Try to get through as many as you can, but you aren't expected to finish everything. Instead, you should make sure everyone in your group knows **how** to solve all the problems, and not just the answers.

Boundedness, Monotonicity & The Squeeze Theorem

1. For each of the following, give an example of sequence with the required properties or explain why no such sequence can exist:
   (a) Bounded, Monotonic, Convergent
   (b) Bounded, Monotonic, Not Convergent
   (c) Bounded, Not Monotonic, Convergent
   (d) Bounded, Not Monotonic, Not Convergent
   (e) Not Bounded, Monotonic, Convergent
   (f) Not Bounded, Monotonic, Not Convergent
   (g) Not Bounded, Not Monotonic, Convergent
   (h) Not Bounded, Not Monotonic, Not Convergent

2. Find $\lim_{n\to\infty} \frac{\cos^2 n}{n}$, if it exists.

3. Find $\lim_{n\to\infty} \frac{n!}{n^n}$, if it exists. Hint: what is the relationship between $\frac{n!}{n^n}$ and $\frac{1}{n}$?

4. (a) Prove that if $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.
   (b) Find a counterexample to the statement "If $\lim_{n\to\infty} |a_n| = L$, then $\lim_{n\to\infty} a_n = L$"

5. Last time, we found limits of recursively defined sequences, **assuming** that the limits existed. Let's see why this can be a dangerous assumption:
   (a) Consider the sequence $a_1 = 1$, $a_n = -a_{n-1}$. Assuming this sequence converges, find its limit.
   (b) Find a closed form for a_n. Does this sequence actually converge?
   (c) Where is the discrepancy?

Induction

1. Consider the sequence $\{a_n\} = \{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots\}$
   (a) Find a recursive definition for a_n.
   (b) Using induction, show that $0 \le a_n \le 2$
   (c) Again using induction, show that $\{a_n\}$ is monotonically increasing.
   Hint: you will probably need to use the result of part (b)
   (d) Show that $\{a_n\}$ is convergent and find its limit if you didn't do so last time.

2. The Fibonacci numbers are a sequence defined by the recurrence relation $F_1 = 1, F_2 = 2, F_n = F_{n-1} + F_{n-2}$
   (a) Write out the first few terms of the fibonacci sequence
   (b) Consider the sequence $a_1 = \frac{1}{2}, a_n = \frac{1}{1+a_{n-1}}$. Write out the first few terms of this sequence. What do you notice?

(c) Use induction to prove that $a_n = \frac{F_n}{F_{n+1}}$

(d) Assuming $\lim_{n\to\infty} a_n$ exists (which it does, but it's a pain to prove!), find $\lim\limits_{n\to\infty} \dfrac{F_n}{F_{n+1}}$

3. Let's find a closed form for the Fibonacci sequence, which is defined in the problem above.

   (a) Use induction to show that if x satisfies the equation $x^2 = x + 1$, then $x^n = xF_n + F_{n-1}$.
   Hint: $x^n = xx^{n-1}$

   (b) Let $y = \frac{-1+\sqrt{5}}{2}, z = \frac{-1-\sqrt{5}}{2}$ be the two roots of $x^2 = x + 1$. From part (a), we know that $y^n = yF_n + F_{n-1}$ and that $z^n = zF_n + F_{n-1}$. Subtract these equations and plug in the values of y and z to find a closed form for F_n.

Extra Problems

1. True/False. For all problems, a_n and b_n are sequences. Justify your answers with a sketch of a proof or a counterexample.

   (a) If a_n and b_n converge, then $a_n + b_n$ converges.

   (b) If $a_n + b_n$ converges, then a_n and b_n converge.

   (c) If a_n and b_n converge, then a_n/b_n converges.

   (d) If a_n and b_n diverge, then $a_n + b_n$ diverges.

   (e) If $a_n + b_n$ diverges, then a_n and b_n diverge.

   (f) If a_n and b_n diverge, then a_nb_n diverges.