Math 1B Discussion Section Problems

Rob Bayer

July 15, 2008

You should work on the following problems in groups of 3 or 4. Try to get through as many as you can, but you aren't expected to finish everything. Instead, you should make sure everyone in your group knows **how** to solve all the problems, and not just the answers.

Second Order Equations

1. For each of the following pairs of functions, determine whether they are linearly independent or dependent:
   (a) $f(x) = e^x,\ g(x) = e^{-x}$
   (b) $f(x) = e^x,\ g(x) = e^{x+1}$
   (c) $f(x) = xe^{2x},\ g(x) = e^{2x}$
   (d) $f(x) = \ln(x^3),\ g(x) = \ln(x^\pi)$
   (e) $f(x) = \sin(x),\ g(x) = \cos(x)$
2. Which of the following second order differential equations are linear? Homogeneous?
   (a) $e^x y'' + \cos(3x^2)y' + 3y = 0$
   (b) $y'' + 3y' + 7y = \cos x$
   (c) $y'' + 3xy' + y^2 = 0$
   (d) $\tan(y'') + \cos(x)y' = e^x$
3. Consider the differential equation $y'' = -y$
   (a) By just thinking about it for a while, come up with two linearly independent solutions to this equation and then use them to find the general solution.
   (b) Show that $y = \cos(x + a)$ is a solution for any constant a.
   (c) Argue that parts (a) and (b) show that $\cos(x + a) = C_1 \cos x + C_2 \sin x$ for an appropriate choice of C_1 and C_2
   (d) Without using trig identities, find C_1 and C_2. (Hint: what should $y(0)$ and $y'(0)$ be? Remember that a is a constant, so C_1 and C_2 can refer to it.) Does this formula look familiar?
4. Most of the time (especially in this class), determining whether two functions are linearly independent can be done by just staring at them and making an educated guess. However, sometimes things are more complicated and sometimes you need actual proof that they are linearly independent. One method for doing this is to compute the Wronskian, $W(f, g) = fg' - f'g$, as it turns out that two functions are linearly dependent if and only if $W(f, g) = 0$.
   (a) Use the Wronskian to show that e^{at} and e^{bt} are linearly independent whenever $a \neq b$
   (Hint: a function $y(x)$ is equal to 0 if and only if $y(x) = 0$ for every value of x)
   (b) Show that e^{at} and te^{at} are linearly independent for any real number a
   (c) Show that $\cos(at)$ and $\sin(at)$ are linearly independent for any real number $a \neq 0$.

Solving Second Order Homogeneous Linear ODEs with Constant Coefficients

1. Find the general solution to $y'' + 3y' - 18y = 0$
2. Solve the initial value problem $y'' + 4y' + 4y = 0$ with $y(0) = 1$ and $y'(0) = 3$
3. Solve the boundary value problem $y'' = y$ with $y(0) = 0$ and $y(2) = 2$
4. (a) Show that if r is a double root of $p(x)$, then r is also a root of $p'(x)$. Hint: r is a double root if and only if $p(x)$ is divisible by $(x - r)^2$.
   (b) Explain how this gives another explanation of why the $(2ar + b)e^{rt}$ you get from plugging in $y = te^{rt}$ is zero when r is a double root.
   (c) It turns out that r is a triple root if and only if r is a root of $p(x), p'(x)$ and $p''(x)$. Use this to show that if r is a triple root of $ar^3 + br^2 + cr + d$, then $e^{rt}, te^{rt}, t^2e^{rt}$ are three linearly independent solutions to $ay''' + by'' + cy' + dy = 0$.