Math 1B Discussion Section SOLUTIONS

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You should work on the following problems in groups of 3 or 4. Try to get through as many as you can, but you aren't expected to finish everything. Instead, you should make sure everyone in your group knows **how** to solve all the problems, and not just the answers.

Separable word problems

1. A certain curve in the plane has the property that **every** normal line (that is, a line perpendicular to the tangent line) to the curve passes through (2,0). Find the equation for this curve if you know it passes through (1,1). Hint: recall that two lines are perpendicular if and only if the product of their slopes is -1.

   We know that

   $$\begin{aligned} y' = \text{slope of tangent line at (x,y)} &= -\frac{1}{\text{slope of normal line at (x,y)}} \\ &= = -\frac{1}{\text{slope of line through (x,y) and (2,0)}} \\ &= -\frac{1}{\frac{y-0}{x-2}} = -\frac{x-2}{y} \end{aligned}$$

   This is separable, so we just solve it as $ydy = (x-2)dx$, so $y^2/2 = x^2/2 - 2x + C$. So $y^2 = x^2 - 4x + C$. Plugging in $y(1) = 1$ gives $1 = 1 - 4 + C$, so $C = 4$ and we get $y^2 = x^2 - 4x + 4 = (x-2)^2$

2. (Adapted from Theo Johnson-Freyd and Stewart) Without accounting for fishing, the Pacific Halibut Fishery can be modeled by the differential equation $y' = ky(M-y)$, where $y(t)$ is the biomass at time t, $M = 8 \times 10^7 kg$ is the carrying capacity, and $k = 8.875 \times 10^{-9}$.

   (a) Suppose that in addition to the natural birth and death of fish, fishing companies also harvest H fish per year. Come up with a new differential equation that models this situation and determine its equilibria.

   Since we remove an additional H fish, it's just $y' = ky(M-y) - H$

   (b) As a general rule, we want fishery populations to remain constant, so the managing agencies allow fisherman to harvest the same number of fish each year as the population would naturally grow on its own. What should H be? Solve this differential equation under these conditions.

   We want the population to be steady, meaning we want $y' = 0$. Thus, we want $H = ky(M-y)$

   (c) Assuming we want to provide the maximum possible profits for fishing companies by allowing as much fishing as possible, what is the optimal population of Halibut?

   We want to pick the y that maximizes $ky(M-Y)$. By differentiating and finding critical points, we see that we want $k(M-y) - ky = 0$. Solving, we see that we want $y = M/2$

   (d) Using similar reasoning as in part (c), at what population does a species that obeys the logistic equation grow most rapidly?

   At half it's carrying capacity.

3. Last time we saw that one model for the fall of a baseball is $v' = g - bv^2$. Solve this equation for v and then use your answer to find an equation for the height of the ball at time t.

   Yuk.

4. The differential equation $\frac{dy}{dx} = ky^{1+a}$ is sometimes called the Doomsday Equation. Here we'll try to figure out why:

(a) Solve this equation in terms of k, a, and an arbitrary constant C. How is this different than the solution to $y' = ky$?

$$\begin{aligned}
\frac{dy}{y^{1+a}} &= kdx \\
-\frac{1}{ay^a} &= kx + C \\
y^a &= \frac{1}{-kax + C} \\
y &= \sqrt[a]{\frac{1}{-kax + C}}
\end{aligned}$$

(b) Now suppose the growth of a population of rabbits follows the differential equation $y' = 2y^{1.01}$ and suppose there are initially only two rabbits. When should we all start running for the hills?

This is just $a = 1/100, k = 2$ in the above, so the general solution is $y = \dfrac{1}{(-\frac{x}{50} + C)^{100}}$

Plugging in $y(0) = 2$ gives $2 = \frac{1}{C^{100}}$, so $C \approx 1$. Then $y = \frac{1}{(-\frac{x}{50}+1)^{100}}$

So at $x = 50$ days, there will be INIFNITELY many rabbits.

First Order ODEs of Homogeneous Type

1. Find the general solution to $x^2y' = y^2 + 3yx + x^2$.

We can re-write this is $y' = \left(\frac{y}{x}\right)^2 + 3\frac{y}{x} + 1$ and make the subsitution $v = \frac{y}{x}$. Then $y = vx$, so $y' = v'x + v$:

$$\begin{aligned}
v'x + v &= v^2 + 3v + 1 \\
v' &= \frac{v^2 + 2v + 1}{x} \\
\frac{dv}{(v+1)^2} &= \frac{dx}{x} \\
-\frac{1}{v+1} &= \ln x + C \\
v + 1 &= -\frac{1}{\ln x + C} \\
v &= -\frac{1}{\ln x + C} - 1
\end{aligned}$$

Since we divided by $(v+1)^2$, we need to check if $v = -1$ is a solution: $v'x + v = 0 + -1 = -1$ and $v^2 + 3v + 1 = 1 - 3 + 1 = -1$, so it is a solution.

Thus, $v = -\frac{1}{\ln x + C} - 1$ or $v = -1$ is the general solution. Then $y = vx$, so

$$y = -\frac{x}{\ln x + C} - x \text{ or } y = -x$$

2. Solve the initial value problem $y' = \tan(y/x) + \dfrac{y}{x}$; $y(1) = \pi/2$

We'll make the same substitution as above:

$$\begin{aligned}
v'x + v &= \tan v + v \\
v' &= \frac{\tan v}{x} \\
\frac{\cos v}{\sin v} dv &= \frac{dx}{x} \\
\ln \sin v &= \ln x + C \\
\sin v &= Cx \\
v &= \arcsin Cx
\end{aligned}$$

Since we divided by $\tan v$, we need to check if $v = 0$ is a solution:

$v'x + v = 0 + 0 = 0$ and $\tan v + v = 0 + 0 = 0$, so it is a solution.

However, we can get $v = 0$ as a solution by just taking $C = 0$

Thus, the general solution is $y = x \arcsin Cx$

To solve for C, we need $\pi/2 = \arcsin C$, so $C = 1$. Thus, $y = x \arcsin x$