Math 1B-4 Midterm 2

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You have until 2pm to complete this test. No calculators, books, notes, or consultation with other members of the class are permitted. Your exam should have 4 pages.

Please box/circle your final answers and show enough work to demonstrate that you know what you are doing. Unsupported or improperly supported answers will receive no credit.

Unless otherwise stated, $y = y(x)$ is a function of the independent variable x.

Name: ____________________________________

1	15	
2	10	
3	15	
4	15	
5	15	
6	15	
7	15	
Total	100	

1. Short answer. You need not show any work for these problems:

   (a) (9 pts) Write the form of a particular solution (not just a first-try guess!) to each of the following ODEs. You need not solve for the actual coefficients, though your answer should use as few undetermined coefficients as necessary. For example, a correct answer for $y'' + y = e^x$ would be $y_p = Ae^x$. $y_p = Ae^x + Be^x$ would be an incorrect answer.

      i. $y'' - y = 3e^{-x} + \cos x + 5$
      $r^2 - 1 = 0$ has solution $r = \pm 1$, so $y_h = C_1 e^x + C_2 e^{-x}$
      $$y_p = Axe^{-x} + B\cos x + C\sin x + D$$

      ii. $y'' + y = x\cos x$
      $y'' + y = 0$ has solution $y_h = C_1\cos x + C_2 \sin x$
      so
      $$y_p = x(Ax + B)\cos x + x(Cx + D)\sin x$$

      iii. $y'' - y' - 6y = x^2 e^x \cos x - 5e^x \sin x$
      $y'' - y' - 6y = 0$ has solution $y = C_1 e^{6x} + C_2 e^{-5x}$, so
      $$y_p = (Ax^2 + Bx + C)e^x \cos x + (Dx^2 + Ex + F)e^x \sin x$$

   (b) (6 pts) Consider the boundary value problem $y'' + y = 0$; $y(0) = 0, y(a) = b$. Give examples of real numbers a, b ($a \neq 0$) such that this boundary value problem has

      i. No solutions
      The general solution is $y = C_1 \cos x + C_2 \sin x$. The condition $y(0) = 0$ means $0 = C_1$, so $y(2\pi) = 1$ would give no solutions since that would require $C_1 = 1$

      ii. Exactly one solution
      The condition $y(\pi/2) = 0$ would force $C_2 = 0$, so the only solution would be $y = 0$

      iii. Infinitely many solutions
      The condition $y(2\pi) = 0$ only require $C_1 = 0$, so C_2 could be anything and thus there would be infinitely many solutions

2. (10 pts) Which direction field corresponds with each differential equation?

   (a) $y' = x + y$: I

   (b) $y' = x^2 \sin y$: III

   (c) $y' = x - y^2$: V

   (d) $y' = y/2 + \sin x$: II

   (e) $y' = y \sin y$: IV

   Note: in my eternal battle with technology, I can't seem to get the actual direction fields to show up in this file. See the website for a separate file containing them.

3. (15 pts) A tank with capacity of 100L initially has 50L of pure water in it. Saltwater containing 1kg/L of salt is then pumped in at a rate of 2L/min. The solution is kept thoroughly mixed and drains from the tank at a rate of 1L/min. How much salt is in the tank when it reaches its 100L capacity?

   We'll say $S(t) =$ kgs of salt at time t.

   Then $S' = rate_{in} - rate_{out} = 2L/min \cdot 1kg/L - 1L/min\frac{S}{50+t} = 2 - \frac{S}{50+t}$

   We re-write this as $S' + \frac{1}{50+t}S = 2$, which is linear.

   $I(x) = e^{\int \frac{1}{50+t}} = e^{\ln(50+t)} = 50 + t$

   Thus $S = \frac{\int IQ}{I} = \frac{\int 100+2t}{50+t} = \frac{100t+t^2+C}{50+t}$

   We know $S(0) = 0$, so we need $0 = \frac{0+0+C}{50+0} = \frac{C}{0}$, so $C = 0$.

The tank will reach its capacity at $t = 50$ minutes, so we just need to compute $S(50)$:

$$S(50) = \frac{100 \cdot 50 + 50^2}{50 + 50} = \frac{100 \cdot 50 + 50^2}{2 \cdot 50} = \frac{100 + 50}{2} = 75kg$$

4. (15 pts) Solve the initial value problem $y'' + 2y' - 3y = 8e^x + 10\cos x;\ y(0) = 0, y'(0) = 0$

   The characteristic equation is $r^2 + 2r - 3 = 0$, which factors as $(r-1)(r+3)$, so $y_h = C_1e^x + C_2e^{-3x}$

   We'll use undetermined coefficients:

$$\begin{aligned}
y_p &= Axe^x + B\cos x + C\sin x \\
y_p' &= Ae^x + Axe^x - B\sin x + C\cos x \\
y_p'' &= 2Ae^x + Axe^x - B\cos x - C\sin x \\
y_p'' + 2y_p' - 3y_p &= (2A + 2A)e^x + (A + 2A - 3A)xe^x + (-B + 2C - 3B)\cos x + (-C - 2B - 3C)\sin x \\
&= 4Ae^x + (2C - 4B)\cos x + (-2B - 4C)\sin x
\end{aligned}$$

   We want this to be $8e^x + 10\cos x$, so we need $A = 2$, $2C - 4B = 10$, $-2B - 4C = 0$. From the last of these, we see that $2C = -B$, so the middle equation become $-5B = 10$, so $B = -2$. Since we know $2C = -B$, we get $C = 1$.

   Then $y = C_1e^x + C_2e^{-3x} + 2xe^x - 2\cos x + \sin x$

   $y(0) = 0$ gives $0 = C_1 + C_2 - 2$, so $C_1 + C_2 = 2$

   $y'(0) = 0$ gives $0 = C_1 - 3C_2 + 2 + 1$, so $C_1 - 3C_2 = -3$.

   Adding 3 times the first to the second gives $4C_1 = 3$, so $C_1 = \frac{3}{4}$ and thus $C_2 = \frac{5}{4}$

   So

$$y = \frac{3}{4}e^x + \frac{5}{4}e^{-3x} + 2xe^x - 2\cos x + \sin x$$

5. (15 pts) By making the substitution $v(x) = y/x^2$, find the general solution to $y' = e^{y/x^2} + 2\frac{y}{x}$

   Hint: think about how can you re-write $2\frac{y}{x}$ in terms of v

   If $v = y/x^2$, then $y = vx^2$, so $y' = v'x^2 + 2xv$. Also, $2\frac{y}{x} = 2vx$, so substituting in, we get

$$v'x^2 + 2vx = e^v + 2vx$$

   After subtracting $2vx$ from both sides, we see that this is separable:

$$\begin{aligned}
v'x^2 &= e^v \\
e^{-v}dv &= \frac{dx}{x^2} \\
-e^{-v} &= -\frac{1}{x} + C \\
e^{-v} &= \frac{1}{x} + C \\
-v &= \ln(1/x + C) \\
v &= -\ln(1/x + C)
\end{aligned}$$

   Since $y = vx^2$, we get that $y = -x^2\ln(\frac{1}{x} + C)$

6. (15 pts) Find the general solution to $y' + \frac{y}{x} = x^4e^xy^4$

   This is Bernoulli, so we start by dividing through by y^4 to get $y'y^{-4} + \frac{1}{x}y^{-3} = x^4e^x$

   Since we divided through be y^4 we also need to check if $y = 0$ is a solution. Plugging into the original equation we get: $0 + \frac{0}{x} \overset{?}{=} x^4e^x0$, which is true, so it **is** a solution.

   We now make the substitution $u = y^{-3}$, so $u' = -3y^{-4}y'$. Plugging back in gives $\frac{u'}{-3} + \frac{1}{x}y = x^4e^x$

   So we multiply through by -3 and see that this is linear with $P = -3/x, Q = -3x^4e^x$

So $I = e^{\int \frac{-3}{x}} = e^{-3\ln x} = x^{-3}$
$u = \frac{-3x^4 e^x x^{-3}}{x^{-3}} = -3x^3 \int xe^x = -3x^3(xe^x - e^x + C)$
Then since $\frac{1}{y^3} = u, y = \sqrt[3]{\frac{1}{u}}$, so

$$y = \sqrt[3]{\frac{1}{-3x^3(xe^x - e^x + C)}} \text{ or } y = 0$$

7. (15 pts) Find the general solution to $y'' + 4y' + 5y = e^{-2x}\tan x$

The characteristic equation is $r^2 + 4r + 5 = 0$, so $r = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$
So $y_h = C_1 e^{-2x}\cos x + C_2 e^{-2x}\sin x$
We're going to have to use variation of parameters here, so let's calculate $y_1y_2' - y_1'y_2$:

$$\begin{aligned}
y_1y_2' - y_1'y_2 &= e^{-2x}\cos x(-2e^{-2x}\sin x + e^{-2x}\cos x) - (-2e^{-2x}\cos x - e^{-2x}\sin x)e^{-2x}\sin x \\
&= -2e^{-4x}\sin x\cos x + e^{-4x}\cos^2 x + 2e^{-4x}\cos x\sin x + e^{-4x}\sin^2 x \\
&= e^{-4x}(\cos^2 x + \sin^2 x) \\
&= e^{-4x}
\end{aligned}$$

Then from the formula sheet, we know

$$\begin{aligned}
u_2 &= \int \frac{e^{-2x}\cos x e^{-2x}\tan x}{e^{-4x}} dx \\
&= \int \cos x \tan x \\
&= \int \sin x \\
&= -\cos x \\
u_1 &= -\int \frac{e^{-2x}\sin x e^{-2x}\tan x}{e^{-4x}} sdx \\
&= -\int \frac{\sin^2 x}{\cos x} \\
&= -\int \frac{1 - \cos^2 x}{\cos x} \\
&= -\int \sec x - \cos x \\
&= -\ln(\sec x + \tan x) + \sin x
\end{aligned}$$

Then

$$\begin{aligned}
y &= C_1e^{-2x}\cos x + C_2e^{-2x}\sin x + u_1y_1 + u_2y_2 \\
&= C_1e^{-2x}\cos x + C_2e^{-2x}\sin x + (\sin x - \ln(\sec x + \tan x))e^{-2x}\cos x - \cos x e^{-2x}\sin x \\
&= C_1e^{-2x}\cos x + C_2e^{-2x}\sin x - \ln(\sec x + \tan x)e^{-2x}\cos x
\end{aligned}$$