Math 1B-4 Midterm 1 SOLUTIONS

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You have until 2pm to complete this test. No calculators, books, notes, or consultation with other members of the class are permitted. Your exam should have 8 pages.

Please box/circle your final answers and show enough work to demonstrate that you know what you are doing. Unsupported answers will receive no credit.

Name: ____________________________________

1	10	
2	15	
3	15	
4	15	
5	15	
6	18	
7	12	
Total	100	

1. Short answer. You need not show any work for these problems:

   (a) (3 pts) Find the form of the partial fraction decomposition for $\dfrac{x^2+3x-1}{x^3(x-1)(x^2+x+1)^2}$.
   You do **not** need to solve for the coefficients

$$\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{x-1}+\frac{Ex+F}{x^2+x+1}+\frac{Gx+H}{(x^2+x+1)^2}$$

   (b) (3 pts) Re-write $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{e^x(x-1)}$ as limits of proper integrals.
   If you like, you may use the shorthand $\dfrac{1}{e^x(x-1)}=f$.

$$\lim_{a\to-\infty}\int_a^0 f+\lim_{b\to1^-}\int_0^b f+\lim_{c\to1^+}\int_c^2 f+\lim_{d\to\infty}\int_2^d f$$

   (c) (1 pt) $\displaystyle\int_0^1\frac{dx}{x^p}$ converges for $\underline{\ p<1\ }$ and diverges for $\underline{\ p\geq 1\ }$

   (d) (1 pt) $\displaystyle\int_1^{\infty}\frac{dx}{x^p}$ converges for $\underline{\ p>1\ }$ and diverges for $\underline{\ p\leq 1\ }$

   (e) (2 pts) The arclength of the curve $y=f(x)$ for $a\leq x\leq b$ is given by $\underline{\displaystyle\int_a^b\sqrt{1+(f')^2}dx}$

2. (15 pts) $\int x^2 \tan^{-1} x dx$

By parts: $\begin{array}{ll} u = \tan^{-1} x & dv = x^2 dx \\ du = \frac{1}{1+x^2} dx & v = \frac{x^3}{3} \end{array}$

$$\begin{aligned} \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int \frac{x^3}{1+x^2} &= \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int x - \frac{x}{1+x^2} dx \\ &= \frac{x^3}{3}\tan^{-1} x - \frac{x^2}{6} + \frac{1}{6}\ln(1+x^2) + C \end{aligned}$$

3. (15 pts) $\int_1^2 \frac{9}{x(x-3)^2}dx$ Hint: this integral **does not** diverge.

$$\begin{aligned}
\frac{9}{x(x-3)^2} &= \frac{A}{x} + \frac{B}{x-3} + \frac{C}{(x-3)^2} \\
9 &= A(x-3)^2 + Bx(x-3) + Cx \\
x = 0 : 9 &= 9A \\
x = 3 : 9 &= 3C \\
x = 4 : 9 &= A + 4B + 4C
\end{aligned}$$

Solving these will give $A = 1, B = -1, C = 3$

$$\begin{aligned}
\int_1^2 \frac{9}{x(x-3)^2}dx &= \int_1^2 \frac{1}{x} - \frac{1}{x-3} + \frac{3}{(x-3)^2}dx \\
&= \ln|x| - \ln|x-3| - \frac{3}{x-3}\Big|_1^2 \\
&= \ln 2 - \ln 1 + 3 - \ln 1 + \ln 2 - \frac{3}{2} \\
&= 2\ln 2 + \frac{3}{2}
\end{aligned}$$

4. (15 pts) $\int \frac{dx}{x^2\sqrt{1+x^2}}$

We'll do $x = \tan\theta$, so $dx = \sec^2\theta d\theta$:

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{1+x^2}} &= \int \frac{\sec^2\theta}{\tan^2\theta\sqrt{1+\tan^2\theta}} d\theta \\
&= \int \frac{\sec\theta}{\tan^2\theta} d\theta = \int \frac{1}{\cos\theta}\frac{\cos^2\theta}{\sin^2\theta} d\theta \\
&= \int \frac{\cos\theta}{\sin^2\theta} d\theta \\
&= \int \frac{1}{u^2} du \\
&= -\frac{1}{u} + C \\
&= -\frac{1}{\sin\theta} + C \\
&= -\frac{\sqrt{1+x^2}}{x} + C
\end{aligned}$$

5. (15 pts) $\displaystyle\int_0^4 \frac{\ln x}{\sqrt{x}} dx$

Because of the $\sqrt{x}$ in the denominator, this is an improper integral whenever 0 is included in our range of integration:

For simplicity, let's start by finding an anti-derivative of $\frac{\ln x}{\sqrt{x}}$ by integrating by parts with $u = \ln x$, $dv = x^{-1/2}$, so $du = 1/x$, $v = 2x^{1/2}$:

$$\begin{aligned}
\int \frac{\ln x}{\sqrt{x}} dx &= 2\sqrt{x}\ln x - 2\int \frac{\sqrt{x}}{x} dx \\
&= 2\sqrt{x}\ln x - 2\int x^{-1/2} dx \\
&= 2\sqrt{x}\ln x - 4\sqrt{x} \text{ (Note that we don't need +C since we'll be taking a definite integral)}
\end{aligned}$$

So now we have:

$$\begin{aligned}
\int_0^4 \frac{\ln x}{\sqrt{x}} dx &= \lim_{t\to 0^+} \int_t^4 \frac{\ln x}{\sqrt{x}} dx \\
&= \lim_{t\to 0^+} 2\sqrt{x}\ln x - 4\sqrt{x}\Big|_t^4 \\
&= \lim_{t\to 0^+} 2\sqrt{4}\ln 4 - 4\sqrt{4} - 2\sqrt{t}\ln t + 4\sqrt{t} \\
&= 4\ln 4 - 8 \text{ (Since } \lim_{t\to 0^+} \sqrt{t}\ln t = 0 \text{ as given on the formulas page)}
\end{aligned}$$

6. (18 pts) Determine whether each improper integral is convergent or divergent. You do **not** need to evaluate them. Note that this problem continues onto the next two pages.

(a) $\displaystyle\int_3^\infty \frac{1+e^{-x}}{\sqrt[3]{x^2-x-2}}dx$

$1+e^{-x} > 1$ and $x^2-x-2 < x^2$, so $\frac{1+e^{-x}}{\sqrt[3]{x^2-x-2}} > \frac{1}{\sqrt[3]{x^2}} = \frac{1}{x^{2/3}}$

Also, $\int_3^\infty \frac{1}{x^{2/3}}dx$ diverges since it is a p-integral with $p = 2/3$

Thus, by the comparison test, the original integral diverges too.

(b) $\displaystyle\int_0^{\pi/2} \frac{\sin x}{\sqrt{x}}dx$

In the interval $(0, \pi/2)$, $0 < \sin x < 1$, so $\frac{\sin x}{\sqrt{x}} < \frac{1}{\sqrt{x}}$

Also, $\int_0^{\pi/2} \frac{1}{\sqrt{x}}$ converges since it is a $0 \to 1$ p-integral with $p = 1/2$

Thus, by the comparison test, the original integral converges too.

(c) $\displaystyle\int_\pi^\infty \sin^3 x \, dx$

$$\begin{aligned}
\int_\pi^\infty \sin^3 x dx &= \lim_{t\to\infty} \int_\pi^t \sin^3 x dx \\
&= \lim_{t\to\infty} \int_\pi^t (1-\cos^2 x)\sin x dx \\
&= \lim_{t\to\infty} -\int_{x=\pi}^t 1-u^2 du \\
&= -\lim_{t\to\infty} \cos x - \frac{\cos^3 x}{3}\bigg|_\pi^t \\
&= -\lim_{t\to\infty} \cos t - \frac{\cos^3 t}{3} - (-1 - \frac{-1}{3})
\end{aligned}$$

But this limit is not defined since cos never approaches a value as $t \to \infty$. Thus, the integral is divergent.

7. (12 pts) Find the surface area of the object obtained by rotating $y = \sqrt{x}$ about the x-axis for $\frac{3}{4} \leq x \leq \frac{15}{4}$
Note that $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

$$
\begin{aligned}
2\pi \int_{3/4}^{15/4} \sqrt{x}\sqrt{1 + \frac{1}{4x}}dx &= 2\pi \int_{3/4}^{15/4} \sqrt{x + \frac{1}{4}}dx \\
&= 2\pi \left(\left. \frac{(x + \frac{1}{4})^{3/2}}{3/2} \right|_{3/4}^{15/4} \right) \\
&= \frac{4\pi}{3}(4^{3/2} - 1^{3/2}) = \frac{28\pi}{3}
\end{aligned}
$$