Math 1B-4 Final Exam

Rob Bayer

August 15, 2008

You have until 2pm to complete this test. No calculators, books, notes, or consultation with other members of the class are permitted. Your exam should have 11 pages.

Please box/circle your final answers and show enough work to demonstrate that you know what you are doing. Unsupported or improperly supported answers will receive no credit. Be sure to **clearly state which theorems or tests you are using**.

Unless otherwise stated, $y = y(x)$ is a function of the independent variable x.

Name: ________________________________

1	10	
2	10	
3	15	
4	15	
5	15	
6	15	
7	10	
8	10	
Total	100	

1. (10 pts) Quick answer. You need not show any work for these problems:

(a) For which values of p do each of the following quantities converge and diverge?

	Converges	Diverges
$\int_1^\infty \frac{dx}{x^p}$	$p > 1$	$p \leq 1$
$\int_0^1 \frac{dx}{x^p}$	$p < 1$	$p \geq 1$
$\sum_{n=1}^\infty \frac{1}{n^p}$	$p > 1$	$p \leq 1$
$\sum_{n=1}^\infty \frac{(-1)^n}{n^p}$	$p > 0$	$p \leq 0$
$\lim_{n\to\infty} \frac{1}{n^p}$	$p \geq 0$	$p < 0$

(b) Complete the following definitions:

i. $\sum a_n$ is conditionally convergent iff[1] $\underline{\sum a_n \text{ converges and } \sum |a_n| \text{ diverges}}$

ii. $\sum a_n$ is absolutely convergent iff $\underline{\sum |a_n| \text{ converges}}$

iii. R is the ROC of $\sum c_n(x-a)^n$ iff $\underline{\sum c_n(x-a)^n \text{ converges when } |x-a| < R \text{ and diverges when } |x-a| > R}$

iv. A first order ODE is linear iff it can be written in the form $\underline{y' + P(x)y = Q(x)}$

v. A first order ODE is separable iff it can be written in the form $\underline{y' = f(x)g(y)}$

---

[1] iff is shorthand for "if and only if"

2. (10 pts) Find the general solution to $xy' + 2y = x \sin x$

After dividing through by x we see that this is linear with $P(x) = \frac{2}{x}$, $Q(x) = \sin x$.

So $I = e^{\int \frac{2}{x}} = e^{2 \ln x} = x^2$

Then we can integrate IQ by parts twice to get:

$$\begin{aligned}
y &= \frac{\int IQ}{I} = \frac{1}{x^2}\left(\int x^2 \sin x\right) \\
&= \frac{1}{x^2}\left(-x^2 \cos x + 2\int x \cos x\right) \\
&= \frac{1}{x^2}\left(-x^2 \cos x + 2x \sin x - 2\int \sin x\right) \\
&= \frac{-x^2 \cos x + 2x \sin x + 2 \cos x + C}{x^2}
\end{aligned}$$

3. (15 pts) Find the general solution to $y' = \dfrac{y^2}{x^2\sqrt{4+x^2}}$

This is separable:

$$\begin{aligned}
\int \frac{dy}{y^2} &= \int \frac{dx}{x^2\sqrt{4+x^2}} \\
-\frac{1}{y} &= \int \frac{2\sec^2\theta d\theta}{4\tan^2\theta\sqrt{4+4\tan^2\theta}} \qquad (x = 2\tan\theta, dx = 2\sec^2\theta) \\
&= \int \frac{\sec\theta d\theta}{4\tan^2\theta} \\
&= \frac{1}{4}\int \frac{\cos^2\theta}{\sin^2\theta}\frac{1}{\cos\theta} \\
&= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta} \\
&= \frac{1}{4}\int \frac{1}{u^2} \qquad (u = \sin\theta, du = \cos\theta) \\
&= \frac{1}{4}\left(-\frac{1}{u} + C\right) \\
&= -\frac{1}{4}\csc\theta + C \\
&= -\frac{\sqrt{4+x^2}}{4x} + C \\
y &= \frac{1}{\frac{\sqrt{4+x^2}}{4x} + C} = \frac{1}{\frac{\sqrt{4+x^2}+Cx}{4x}} = \frac{4x}{\sqrt{4+x^2}+Cx}
\end{aligned}$$

Since we divided by y^2 at the very start, we also need to check $y = 0$. Plugging this into the original equation shows that it is a solution so the final answer is:

$$y = \frac{4x}{\sqrt{4+x^2}+Cx} \text{ or } y = 0$$

4. (15 pts) Find the general solution to $y'' + 4y' + 4y = \frac{e^{-2x}}{x^3}$

The characteristic equation is $r^2 + 4r + 4 = 0$, which factors as $(r+2)^2 = 0$.

So $y_1 = e^{-2x}, y_2 = xe^{-2x}$

The RHS is not something we know how to do undetermined coefficients for, so we'll do variation of parameters:

$$\begin{aligned}
y_1y_2' - y_1'y_2 &= e^{-2x}(e^{-2x} - 2xe^{-2x}) + 2e^{-2x}xe^{-2x} \\
&= e^{-4x} \\
u_1 &= -\int \frac{xe^{-2x}\frac{e^{-2x}}{x^3}}{e^{-4x}} \\
&= -\int \frac{1}{x^2} \\
&= \frac{1}{x} \\
u_2 &= \int \frac{e^{-2x}\frac{e^{-2x}}{x^3}}{e^{-4x}} \\
&= \int \frac{1}{x^3} \\
&= -\frac{1}{2x^2}
\end{aligned}$$

So

$$y = C_1e^{-2x} + C_2xe^{-2x} + \frac{e^{-2x}}{x} - \frac{xe^{-2x}}{2x^2} = e^{-2x}\left(C_1 + C_2x + \frac{1}{2x}\right)$$

5. (15 pts) Determine whether each of the following are Conditionally Convergent, Absolutely Convergent, or Divergent. You must clearly state which tests you are using:

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

   $\frac{1}{n \ln n}$ is decreasing and has the limit 0, so by the AST, this series converges.

   $\displaystyle\sum |a_n| = \sum \frac{1}{n \ln n}$

   These terms are positive and decreasing, so we'll do the integral test:

   $\int_2^{\infty} \frac{dx}{x \ln x} = \int_{\ln 2}^{\infty} \frac{du}{u}$, which is a divergent p-integral.

   Thus, by the integral and alternating series tests, $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ is conditionally convergent.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin\left(\frac{1}{n^3}\right)}{\ln\left(1+\frac{1}{n}\right)}$

From the Taylor series on the formula sheet with $x = \frac{1}{n^3}$ and $x = \frac{1}{n}$, we see that this sum is the same as

$$\sum \frac{\frac{1}{n^3} - \frac{1}{3!n^9} + \frac{1}{5!n^{15}} + \cdots}{\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \cdots}$$

Since this series has only positive terms, we'll do the limit comparison test with $\frac{1/n^3}{1/n} = \frac{n}{n^3}$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim \frac{\frac{1}{n^3} - \frac{1}{3!n^9} + \frac{1}{5!n^{15}} + \cdots}{\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \cdots} \frac{n^3}{n} = \lim \frac{1 - \frac{1}{3!n^6} + \frac{1}{5!n^{12}} + \cdots}{1 - \frac{1}{2n} + \frac{1}{3n^2} + \cdots} = \frac{1+0+\cdots}{1+0+\cdots} = 1$$

Since we know $\sum \frac{n}{n^3} = \sum \frac{1}{n^2}$ converges, by the limit comparison test $\displaystyle\sum_{n=1}^{\infty} \frac{\sin\left(\frac{1}{n^3}\right)}{\ln\left(1+\frac{1}{n}\right)}$ does too. Since this series has only positive terms, taking absolute values doesn't change anything, so it is absolutely convergent.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n + 2}{n}$

$\frac{2+\sin n}{n} > \frac{1}{n}$ and $\sum \frac{1}{n}$ diverges ($p = 1$), so by the comparison test, $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n + 2}{n}$ diverges.

6. (15 pts) Solve the initial value problem $y'' + xy' + y = 0;\ y(0) = 1, y'(0) = 0$.
Your answer may include power series if you wish.

Let $y = c_0 + c_1x + c_2x^2 + \cdots$. Then our initial conditions tell us that we need $c_0 = 1, c_1 = 0$.

If $y = \sum_{n=0}^{\infty} c_n x^n$, then $xy' = \sum_{n=1}^{\infty} nc_nx^n = \sum_{n=0}^{\infty} nc_nx^n$
and $y'' = \sum_{n=2}^{\infty} n(n-1)c_nx^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$

Plugging all this in to the differential equation gives:

$$\sum_{n=0}^{\infty}((n+2)(n+1)c_{n+2} + nc_n + c_n)x^n = 0$$

Thus, we need $c_{n+2} = -\frac{c_n}{n+2}$

$$\begin{array}{ll} c_2 = -\frac{c_0}{2} = -\frac{1}{2} & c_3 = -\frac{c_1}{3} = 0 \\ c_4 = -\frac{c_2}{4} = \frac{1}{2\cdot 4} & c_5 = -\frac{c_3}{5} = 0 \\ c_6 = -\frac{c_4}{6} = -\frac{1}{2\cdot 4\cdot 6} & c_7 = 0 \end{array}$$

In general, we see that $c_{2n} = \frac{(-1)^n}{(2n)(2n-2)\cdots 4\cdot 2}$ and $c_{2n+1} = 0$

$$\text{So } y = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)(2n-2)\cdots 4\cdot 2} \underbrace{= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!} = \sum_{n=0}^{\infty} \frac{(-x^2/2)^n}{n!} = e^{-x^2/2}}_{\text{completely optional, but pretty cool...}}$$

7. (10 pts) Find the Taylor series for $f(x) = \frac{1}{x}$ centered at 2.

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	x^{-1}	$\frac{1}{2}$
1	$-x^{-2}$	$-\frac{1}{2^2}$
2	$2x^{-3}$	$\frac{2}{2^3}$
3	$-2 \cdot 3x^{-4}$	$-\frac{2 \cdot 3}{2^4}$
4	$2 \cdot 3 \cdot 4x^{-5}$	$\frac{2 \cdot 3 \cdot 4}{2^5}$

In general, we see that $f^{(n)}(2) = (-1)^n \frac{n!}{2^{n+1}}$

Thus, the taylor series for $\frac{1}{x}$ is

$$\sum_{n=0}^{\infty} (-1)^n \frac{n!}{n! 2^{n+1}} (x-2)^n = \sum_{n=0}^{\infty} (-1)^n \frac{(x-2)^n}{2^{n+1}}$$

8. (10 pts) Find the radius and interval of convergence of $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n(2x-1)^n}{(n+1)(n+2)3^n}$

We'll start with the ratio test:

$$\lim_{n\to\infty}\left|\frac{(n+1)(2x-1)^{n+1}}{(n+2)(n+3)3^{n+1}}\cdot\frac{(n+1)(n+2)3^n}{n(2x-1)^n}\right| = \lim\left|\frac{(n+1)^2(2x-1)}{3n(n+3)}\right| = \frac{|2x-1|}{3}$$

So we know it converges when

$$\begin{aligned}-3 < 2x-1 &< 3\\ -2 < 2x &< 4\\ -1 < x &< 2\end{aligned}$$

And diverges when $x < -1$ or $x > 2$, so we just need to check the endpoints:

$x = -1$: $\sum(-1)^n\frac{n(-3)^n}{(n+1)(n+2)3^n} = \sum\frac{n}{(n+1)(n+2)}$.

$\lim\frac{n}{(n+1)(n+2)}\frac{n^2}{n} = \lim\frac{n^3}{n(n+1)(n+2)} = 1$, and $\sum\frac{n}{n^2} = \sum\frac{1}{n}$ diverges, so by the limit comparison test, the power series diverges at $x = -1$

$x = 2$: $\sum(-1)^n\frac{n3^n}{(n+1)(n+2)3^n} = \sum(-1)^n\frac{n}{(n+1)(n+2)}$ which converges by the AST since the individual terms are decreasing and go to 0.

Thus

$$I = (-1, 2], R = \frac{2-(-1)}{2} = \frac{3}{2}$$