8. $y'' \cancel{\ast} - xy = 0$

$$y = \sum_{n=0}^{\infty} c_n x^n \Rightarrow xy = \sum_{n=0}^{\infty} c_n x^{n+1} = \sum_{n=1}^{\infty} c_{n-1} x^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$$
$$= 2c_2 + \sum_{n=1}^{\infty} (n+2)(n+1) c_{n+2} x^n$$

want: $y'' - xy = 0$

$$2c_2 + \sum_{n=1}^{\infty} \left((n+2)(n+1) c_{n+2} - c_{n-1}\right) x^n = 0$$

$$2c_2 = 0 \quad \text{and} \quad c_{n+2} = \frac{c_{n-1}}{(n+2)(n+1)}$$

$$c_3 = \frac{c_0}{3\cdot 2} \qquad c_4 = \frac{c_1}{4\cdot 3} \qquad c_5 = \frac{c_2}{5\cdot 4} = 0$$

$$c_6 = \frac{c_3}{6\cdot 5} = \frac{c_0}{6\cdot 5\cdot 3\cdot 2} \qquad c_7 = \frac{c_4}{7\cdot 5\cdot} = \frac{c_1}{7\cdot 5\cdot 4\cdot 3} \qquad c_8 = \frac{c_5}{8\cdot 7} = 0$$

In general, $c_{3n} = \dfrac{c_0}{(3n)(3n-1)(3n-3)(3n-4)\cdots 6\cdot 5\cdot 3\cdot 2}$

$$c_{3n+1} = \frac{c_1}{(3n+1)(3n)(3n-2)(3n-3)\cdots 7\cdot 5\cdot 4\cdot 3}$$

$$c_{3n+2} = 0$$

$$y = c_0 \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)(3n-1)\cdots 6\cdot 5\cdot 3\cdot 2} + c_1 \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)(3n)\cdots 7\cdot 5\cdot 4\cdot 3}$$

10. $y'' + x^2 y = 0 \qquad y(0)=1,\ y'(0)=0$

$$y = c_0 + c_1 x + c_2 x^2 + \dots = \sum_{n=0}^{\infty} c_n x^n$$

$\underline{y(0)=1}$: $1 = c_0 + c_1 \cdot 0 + c_2 \cdot 0 + \dots$

$\Rightarrow c_0 = 1$

$\underline{y'(0)=0}$: $0 = c_1 + 2c_2 \cdot 0 + 3c_3 0^2 + \dots$

$\Rightarrow c_1 = 0$

$$x^2 y = \sum_{n=0}^{\infty} c_n x^{n+2} = \sum_{n=2}^{\infty} c_{n-2} x^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$$

$$= 2c_2 + 6c_3 x + \sum_{n=2}^{\infty} (n+2)(n+1) c_{n+2} x^n$$

$\underline{\text{want}}$: $\underbrace{2c_2 + 6c_3 x}_{\downarrow} + \sum_{n=2}^{\infty} \underbrace{\left((n+2)(n+1)c_{n+2} + c_{n-2}\right)}_{\downarrow} x^n = 0$

$2c_2 = 6 \cdot c_3 = 0$

$$c_{n+2} = -\frac{c_{n-2}}{(n+2)(n+1)}$$

$c_4 = -\frac{c_0}{4 \cdot 3} \qquad c_5 = -\frac{c_1}{5 \cdot 4} = 0 \qquad c_6 = -\frac{c_2}{6 \cdot 5} = 0 \qquad c_7 = -\frac{c_3}{7 \cdot 6} = 0$

$c_8 = -\frac{c_4}{8 \cdot 7} = \frac{1}{8 \cdot 7 \cdot 4 \cdot 3} \qquad c_9 = 0 \qquad c_{10} = 0 \qquad c_{11} = 0$

so $c_{4n} = \dfrac{(-1)^n}{(4n)(4n-1) \cdots 8 \cdot 7 \cdot 4 \cdot 3}$ all others $= 0$.

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(4n)(4n-1)(4n-4)(4n-5) \cdots 8 \cdot 7 \cdot 4 \cdot 3}$$