Math 54 Discussion Section SOLUTIONS

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You should work on the following problems in groups of 3 or 4. Try to get through as many as you can, but you aren't expected to finish everything. In fact, the answers are largely unimportant; making sure **everyone** in your group knows **how** to solve all the problems is what really matters.

1. Find a formal solution to the vibrating string problem

$$\begin{aligned} u_{tt} &= 4u_{xx} + x^2 \\ u(0,t) = u(1,t) &= 0 \\ u(x,0) &= 0 \\ u_x(x,0) &= x \end{aligned}$$

   SOLUTION: No solution provided.

2. Let $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$. Find A^{-1}

   SOLUTION: A quick check shows that the columns of this matrix are orthonormal, so $A^{-1} = A^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

3. Without using the methods of section 9.5, solve the system of differential equations $\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \mathbf{y}$

   SOLUTION: This represents the system of differential equations $\begin{cases} y_1' &= y_2 \\ y_2' &= -y_1 - 2y_2 \end{cases}$

   Substituting $y_2 = y_1'$ into the second equation gives $y_1'' + 2y_1' + y_1 = 0$, which we know has solution $y_1 = C_1e^{-t} + C_2te^{-t}$. Then $y_2 = y_1' = -C_1e^t + C_2e^{-t} - C_2te^{-t}$

   So $\mathbf{y} = \begin{bmatrix} C_1e^{-t} + C_2te^{-t} \\ -C_1e^t + C_2e^{-t} - C_2te^{-t} \end{bmatrix} = C_1e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2e^{-t}\begin{bmatrix} t \\ 1-t \end{bmatrix}$

4. Find all eigenvalues and eigenvectors of $L[y] := y'' + 2y'$ over the vector space $V = \{f \in C^\infty : f(0) = f(2\pi) = 0\}$

   We want all the values λ such that $L[y] = \lambda y$ has a non-trivial solution in V. Rephrasing this, we want to find solutions to $y'' + 2y' = \lambda y; y(0) = y(2\pi) = 0, y \neq 0$

   This has characteristic equation $r^2 + 2r - \lambda = 0$, so the discriminant is $4 + 4\lambda$ and $r = \frac{-2\pm\sqrt{4+4\lambda}}{2} = -1 \pm \sqrt{1+\lambda}$

   We go by cases:

   - (2 real roots) This will happen when $1 + \lambda > 0$ and we get $y = C_1e^{(1+\sqrt{1+\lambda})t} + C_2e^{(1-\sqrt{1+\lambda})t}$
     Our boundary conditions become $0 = C_1 + C_2$ and $0 = C_1e^{(1+\sqrt{1+\lambda})2\pi} + C_2e^{(1-\sqrt{1+\lambda})2\pi}$ Playing around with the algebra gives $C_1 = C_2 = 0$, so $y = 0$ is the only solution here. Thus, no $\lambda > -1$ is an eigenvalue.

- (1 repeated root) This happens when $1+\lambda=0$ and we get $r=-1$, so $y=C_1e^{-t}+C_2te^{-t}$
  Our boundary conditions gives $0=C_1$, $0=C_1e^{-2\pi}+C_2 2\pi e^{-2\pi}=C_2 2\pi e^{-2\pi}$ so $C_1=C_2=0$. Thus, $\lambda=-1$ is not an eigenvalue.
- (complex roots) This happens when $1+\lambda<0$. That is, when $\lambda<-1$ and $r=-1\pm i\sqrt{-(1+\lambda)}$, so $y=C_1\cos(-1+\sqrt{-1-\lambda})t+C_2\sin(-1+\sqrt{-1-\lambda})t$
  Our first boundary conditions gives that $0=C_1$, and combining this with our second gives that $0=C_2\sin(-1+\sqrt{-1-\lambda})2\pi$
  Since we are looking for non-trivial solutions, we can assume $C_2\neq 0$ and thus we must have $\sin(-1+\sqrt{-1-\lambda})2\pi=0$
  This only happens when $(-1+\sqrt{-1-\lambda})2\pi=n\pi$ for some integer n, so we must have $-1-\lambda=\left(\frac{n+2}{2}\right)^2$, so $\lambda=-1-\left(\frac{n+2}{2}\right)^2$ is an eigenvalue for any integer n.
  The corresponding eigenfunctions (or eigenvectors, if you prefer) are $y=C\sin\frac{n}{2}t$

5. Consider $\mathbb{P}_3$ with the inner product $\langle p,q\rangle=p(-1)q(-1)+p(0)q(0)+p(1)q(1)$[1]. Find the linear function f that minimizes $\|f-t^3\|$

   We want the closest function in the subspace of $\mathbb{P}_3$ consisting of all functions of the form $at+b$ to the function t^3. This means we want $\text{Proj}_{\text{span}(1,t)}t^3$

   Note that $\{1,t\}$ is already an orthogonal basis since $\langle 1,t\rangle=1(-1)+1(0)+1(1)=0$

   Then $f=\text{Proj}_{\text{span}(1,t)}t^3=\frac{\langle 1,t^3\rangle}{\langle 1,1\rangle}1+\frac{\langle t,t^3\rangle}{\langle t,t\rangle}t=\frac{1(-1)+1(0)+1(1)}{1(1)+1(1)+1(1)}1+\frac{(-1)(-1)+0(0)+1(1)}{(-1)(-1)+0(0)+1(1)}t=t$

6. Find the Fourier sine series of $f(x)=|x|$ on $(0,1)$

   SOLUTION: omitted

[1]Note: This is not actually an inner product on $\mathbb{P}_3$. You would need another point of evaluation to make it into one. Sorry about that