Math 54 Discussion Section SOLUTION

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1. Find a formal solution to the heat equation

$$\begin{aligned} u_t &= u_{xx} + e^{-x} && (1) \\ u(0,t) = u(\pi,t) &= 0 && (2) \\ u(x,0) &= \sin 2x && (3) \end{aligned}$$

Since this is non-homogeneous, we'll make the assertion that $u(x,t) = w(x,t) + v(x)$ where w satisfies

$$\begin{aligned} w_t &= w_{xx} && (4) \\ w(0,t) = w(\pi,t) &= 0 && (5) \end{aligned}$$

and then solve for w and v:
If $u(x,t) = w(x,t) + v(x)$, then

$$\begin{aligned} u_t(x,t) &= w_t(x,t) \text{ (since } \frac{\partial}{\partial t}v(x) = 0) && (6) \\ u_{xx}(x,t) &= w_{xx}(x,t) + v''(x) \text{ (we write } v''(x) \text{ for } v_{xx}) && (7) \\ u(0,t) &= w(0,t) + v(0) && (8) \\ u(\pi,t) &= w(\pi,t) + v(\pi) && (9) \end{aligned}$$

Then plugging (6) - (9) into (1) and (2) gives

$$\begin{aligned} w_t &= w_{xx} + v''(x) + e^{-x} && (10) \\ w(0,t) + v(0) &= 0 && (11) \\ w(\pi,t) + v(\pi) &= 0 && (12) \end{aligned}$$

Now we can use (4) and (5) to get that we must have

$$\begin{aligned} 0 &= v''(x) + e^{-x} && (13) \\ v(0) &= 0 && (14) \\ v(\pi) &= 0 && (15) \end{aligned}$$

Since $v'' = -e^{-x}$, we integrate twice to get $v(x) = -e^{-x} + Cx + D$. By using (14) and (15), we get that $D = 1$ and $C = \frac{e^{-\pi}-1}{\pi}$, so

$$v(x) = -e^{-x} + \frac{e^{-\pi} - 1}{\pi}x + 1 \qquad (16)$$

Now we need to find w. Since w must satisfy (4) and (5), we know that

$$w(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2 t} \sin nx \tag{17}$$

since (4) and (5) are just the homogeneous heat equations from section 10.2 with $\beta = 1, L = \pi$.

To solve for c_n's, we use the fact that $u(x,0) = w(x,0) + v(x)$ must be $\sin 2x$. This is the same as saying $w(x,0) = \sin 2x - v(x)$. Plugging in $t = 0$ and using (16), we get that we must have

$$\sum_{n=1}^{\infty} c_n \sin nx = \sin 2x + e^{-x} - \frac{e^{-\pi} - 1}{\pi} x - 1 \tag{18}$$

This is just the Fourier sine series of the RHS, so we know that

$$c_n = \frac{2}{\pi} \int_0^{\pi} \left(\sin 2x + e^{-x} - \frac{e^{-\pi} - 1}{\pi} x - 1 \right) \sin nx \tag{19}$$

$$= \frac{2}{\pi} \left(e^{-\pi} \frac{(-1)^n - e^{\pi}}{n + n^3} + \left\{ \begin{array}{ll} 0 & n \neq 2 \\ \frac{\pi}{2} & n = 2 \end{array} \right. \right) \tag{20}$$

So our solution (in all its disgusting glory) is

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{\pi} \left(e^{-\pi} \frac{(-1)^n - e^{\pi}}{n + n^3} + \left\{ \begin{array}{ll} 0 & n \neq 2 \\ \frac{\pi}{2} & n = 2 \end{array} \right. \right) e^{-n^2 t} \sin nx \right) + -e^{-x} + \frac{e^{-\pi} - 1}{\pi} x + 1$$

2. Consider the PDE $u_t + tu = u_{xx}$ with boundary conditions $u_x(0,t) = u_x(\pi,t) = 0$.

   (a) Use the method of separation of variables to find all solutions of the form $u(x,t) = X(x)T(t)$.

   (b) Find a solution satisfying the initial condition $u(x,0) = \sin^2 x$

   a) Let's assume $u(x,t) = X(x)T(t)$. Then $u_t = X(x)T'(t), u_{xx} = X''x(x)T(t)$. Plugging these into our original equation gives

$$X(x)T'(t) + tX(x)T(t) = X''(x)T(t) \tag{21}$$

$$\frac{T'(t) + tT(t)}{T(t)} = \frac{X''(x)}{X(x)} \tag{22}$$

Since all the t's are on the left and all the x's are on the right, we know that both of these must be equal to some constant K

Our original boundary conditions give us that $X'(0)T(t) = X'(\pi)T(t) = 0$. So we must either have $T(t) = 0$ or $X'(0) = X'(\pi) = 0$. In the former case, we get that $u(x,t) = 0$ for all x, t, which is boring, so we will consider the case with $X'(0) = X'(\pi) = 0$.

So our goal for the moment is to find non-trivial (ie, non-zero) solutions to $X'' = KX$ with $X'(0) = X'(\pi) = 0$.

This has characteristic equation $r^2 = K$. We now consider 3 cases:

- (2 real roots) This will happen when $K > 0$ and the roots will be $r = \pm\sqrt{K}$, so the solution will be $X(x) = C_1 e^{\sqrt{K}x} + C_2 e^{-\sqrt{K}x}$. We now use $X'(0) = X'(\pi) = 0$ to get that $C_1 = C_2 = 0$, so there are no non-trivial solutions in this case.
- (1 real root) This will happen when $K = 0$, so $r = 0$ is a repeated root and $X(x) = C_1 + C_2 x$. Our boundary conditions give that $C_2 = 0$, but C_1 can be anything, so $K = 0$ **is** an e-val with corresponding e-function $X(x) = C$

- (2 cpx roots) This happens when $K < 0$ and the roots are $r = \pi i\sqrt{-K}$, so $X(x) = C_1 \cos\sqrt{-K}x + C_2 \sin\sqrt{-K}x$

  Then $X'(x) = -C_1\sqrt{-K}\sin\sqrt{-K}x + C_2\sqrt{-K}\cos\sqrt{-K}x$ so our boundary conditions are $C_2\sqrt{-K} = 0$ and $-C_1\sqrt{-K}\sin\sqrt{-K}\pi = 0$ From these we conclude that $C_2 = 0$ and that in order to have a non-trivial solution we must have $\sin\sqrt{-K}\pi = 0$. This can only happen when $\sqrt{-K}\pi = n\pi$ for some integer n, so $K = -n^2$ are the eigenvalues and the corresponding eigenfunctions are $X_n(x) = C_n \cos nx$ for any $n > 0$

  Note: The $K = 0$ case happens to also work if we just plug in $n = 0$ to the $K < 0$ case, so we will slightly abuse our notation and say that $K = -n^2$ are the eigenvalues and $X_n(x) = C_n \cos nx$ are the corresponding eigenfunctions for any $n \geq 0$

  We must now find T_n by solving $T'(t) + tT(t) = -n^2T(t)$ for each $n \geq 0$. Note that this is a separable equation since we can re-write it as

$$\frac{dT_n}{T_n} = (-t - n^2)dt \tag{23}$$

$$\ln T_n = -\frac{t^2}{2} - n^2t + C \tag{24}$$

$$T_n = Ce^{-t^2/2 - n^2t} \tag{25}$$

Combining these gives that $u_n(x,t) = c_n e^{-t^2/2-n^2t}\cos nx$, so the general solution is the sum of these:

$$u(x,t) = \sum_{n=0}^{\infty} c_n e^{-t^2/2-n^2t}\cos nx$$

b) We want $u(x, 0) = \sin^2 x$, so we need to pick our c_n's such that $\sum_{n=0}^{\infty} c_n \cos nx = \sin^2 x$

If we slightly tweak the way we write the sum by pulling out the $n = 0$ term and dividing it by 2, we need $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \sin^2 x$, so we're just looking for the cosine series of $\sin^2 x$. Details omitted.