Math 54 Discussion Section Problems

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Don't freak out. Some of these problems are meant to be hard.

1. Prove that if A is invertible, then AB is similar to BA
2. Let $W = \text{span}\ \{(1,0,1,-1)^T, (0,1,2,2)^T\}$. Find an orthogonal basis for $W^\perp$
3. Let A be a square matrix. Prove that A is invertible if and only if 0 is not an eigenvalue of A by showing:
   (a) If 0 is not an eigenvalue of A, then A is invertible.
   (b) If A is invertible, then 0 is not an eigenvalue of A.
4. Prove or give a counterexample:
   (a) If λ is an eigenvalue of A with corresponding eigenvector $\mathbf{v}$, then $\mathbf{v}$ is an eigenvector of A^n for any positive integer n.
   (b) If λ is an eigenvalue of A with corresponding eigenvector $\mathbf{v}$, then $\mathbf{v}$ is an eigenvector of A^{-1}.
5. Find the eigenvalues of A^9, where $A = \begin{bmatrix} 1 & 3 & 7 & 11 \\ 0 & -1 & 3 & 8 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$
6. Compute A^{10}, where $A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$
7. Give an example of each of the following:
   (a) A matrix that is invertible, but not diagonalizable
   (b) A matrix that is diagonalizable, but not invertible
   (c) A matrix that is diagonalizable, but not orthogonally diagonalizable
   (d) A matrix that is diagonalizable, but has no QR factorization
8. Find the eigenvalues and eigenvectors of $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$. Are matrices of this form always diagonalizable?
9. Let $V = C[-1,1]$ and give it the standard inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$ Find the degree-2 polynomial $p(x)$ that minimizes $\|e^x - p\|$
10. Suppose A is a 3×3 matrix with eigenvalues 1 and 2 and that $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvalue corresponding to $\lambda = 1$ and that $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$ are eigenvalues corresponding to $\lambda = 2$. Find A.