Math 54 Discussion Section SOLUTIONS

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Don't freak out. Some of these problems are meant to be hard.

1. Prove that if A is invertible, then AB is similar to BA

   $AB = A(BA)A^{-1}$, so we can just use $P = A$ in the definition of similar.

2. Let $W = \text{span}\ \{(1,0,1,-1)^T, (0,1,2,2)^T\}$. Find an orthogonal basis for $W^\perp$

   We know $(\text{Col } A)^\perp = \text{Nul } A^T$, so we just compute Nul $\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 2 \end{bmatrix}$ and get that $\{(-1,-2,1,0)^T, (1,-2,0,1)^T\}$ is a basis for $W^\perp$. To make it orthogonal, use gram-schmidt: (details omitted).

3. Let A be a square matrix. Prove that A is invertible if and only if 0 is not an eigenvalue of A by showing:

   A cuter way to do this problem would be to say:

   0 is an eigenvalue $\Leftrightarrow \det(A - 0I) = 0 \Leftrightarrow \det A = 0 \Leftrightarrow A$ is not invertible.

   (a) If 0 is not an eigenvalue of A, then A is invertible.
   If 0 is not an eigenvalue, then $Ax = 0x$ has only the trivial solution, so A is invertible.

   (b) If A is invertible, then 0 is not an eigenvalue of A.
   If A is invertible, then A has trivial nullspace, so $Ax = 0$ has only the trivial solution. But this equation is the same as $Ax = 0x$, so we see that there are no eigenvectors corresponding to $\lambda = 0$

4. Prove or give a counterexample:

   (a) If λ is an eigenvalue of A with corresponding eigenvector $\mathbf{v}$, then $\mathbf{v}$ is an eigenvector of A^n for any positive integer n.
   $A^n v = \underbrace{A \cdot A \cdots A}_{n} v = \underbrace{A \cdot A \cdots A}_{n-1}(Av) = \underbrace{A \cdot A \cdots A}_{n-1} \lambda v = \lambda \underbrace{A \cdot A \cdots A}_{n-2} Av = \lambda^2 \underbrace{A \cdot A \cdots A}_{n-3}(Av) = \cdots = \lambda^n v$
   so v is an eigenvector of A corresponding to the eigenvalue λ^n

   (b) If λ is an eigenvalue of A with corresponding eigenvector $\mathbf{v}$, then $\mathbf{v}$ is an eigenvector of A^{-1}.
   Since $Av = \lambda v$, we know $A^{-1}Av = \lambda A^{-1}v$. Since A is invertible, we know $\lambda \neq 0$, so we can divide both sides of the equation to get $\frac{1}{\lambda}v = A^{-1}v$, so v is an e-vec corresponding to the eigenvalue $\frac{1}{\lambda}$

5. Find the eigenvalues of A^9, where $A = \begin{bmatrix} 1 & 3 & 7 & 11 \\ 0 & -1 & 3 & 8 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

   The eigenvalues of A are $1, -1, -2, 2$, so the eigenvalues of A^9 are $1^9, (-1)^9, (-2)^9, 2^9$.

6. Compute A^{10}, where $A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$

   The eigenvalues of A are 1, 2 with corresponding eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

   So $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ (the last matrix here is just the inverse of the first one)

   Then $A^{10} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1^{10} & 0 \\ 0 & 2^{10} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1023 & 1024 \end{bmatrix}$

7. Give an example of each of the following:

   (a) A matrix that is invertible, but not diagonalizable
   $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

   (b) A matrix that is diagonalizable, but not invertible
   $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

   (c) A matrix that is diagonalizable, but not orthogonally diagonalizable
   $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$
   Note: this is basically just asking for a non-symmetric diagonalizable matrix.

   (d) A matrix that is diagonalizable, but has no QR factorization
   $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

8. Find the eigenvalues and eigenvectors of $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$. Are matrices of this form always diagonalizable?

   $\det \begin{bmatrix} a-\lambda & b \\ b & a-\lambda \end{bmatrix} = (a-\lambda)^2 - b^2$, so we need $a - \lambda = \pm b$. That is, $\lambda = a \pm b$ are the eigenvalues.

   If $b = 0$, then our matrix is already diagonal and $A - aI = \begin{bmatrix} 0 & 0 \\ 0 & 0\lambda \end{bmatrix}$, so every vector is an eigenvector of A. A basis for the eigenspace is thus e_1, e_2.

   So now let's look at $b \neq 0$. Then $a + b \neq a - b$, so A is diagonalizable since it has two distinct eigenvalues. Let's find bases for each of the eigenspaces:

   $(\lambda = a+b)$: $A - \lambda I = \begin{bmatrix} -b & b \\ b & -b \end{bmatrix}$. So a basis for the eigenspace is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

   $(\lambda = a-b)$: $A - \lambda I = \begin{bmatrix} b & b \\ b & b \end{bmatrix}$. So a basis for the eigenspace is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

   Note: regardless of whether $b = 0$, we can find two linearly indep. eigenvectors, so all matrices of this form are diagonalizable.

9. Let $V = C[-1, 1]$ and give it the standard inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$ Find the degree-2 polynomial $p(x)$ that minimizes $\|e^x - p\|$

   This problem is a pain. In short: start by using G-S to find an orthogonal basis for span $\{1, t, t^2\}$. You should get $\{1, t, t^2 - \frac{1}{3}\}$. Then compute the projection of e^x onto that basis and get $\frac{e + \frac{1}{e}}{2} + \frac{3}{e}t + \frac{15(e^2-7)}{4e}(t^2 - \frac{1}{3})$

10. Suppose A is a 3×3 matrix with eigenvalues 1 and 2 and that $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvalue corresponding to $\lambda = 1$ and that $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$ are eigenvalues corresponding to $\lambda = 2$. Find A.

    $$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix}^{-1} = \text{messy calculations}$$