Name: ______________________________
Section: 9-10 11-12 2-3

Math 54 Quiz 9 SOLUTIONS

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You have 20 minutes to complete this quiz. You must show your work.

1. (2 pts)

   (a) Define what it means for $\widehat{\mathbf{x}}$ to be a least squares solution to $A\mathbf{x} = \mathbf{b}$. Do not use any definition that includes A^T

   $\|A\widehat{\mathbf{x}} - \mathbf{b}\| \leq \|A\mathbf{x} - \mathbf{b}\|$ for all $\mathbf{x} \in \mathbb{R}^n$

   (b) Complete the following definition: $\langle \cdot, \cdot \rangle$ is an inner product for V if for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and all $c \in \mathbb{R}$:

   i. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$

   ii. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$

   iii. $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$

   iv. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$, and $\langle \mathbf{u}, \mathbf{u} \rangle = 0 \Rightarrow \mathbf{u} = 0$

2. (a) (3 pts) Find all least squares solutions to $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix}$

   We need to solve $A^T A\mathbf{x} = A^T\mathbf{b}$

   $$A^T A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \quad A^T\mathbf{b} = \begin{bmatrix} 14 \\ 4 \\ 10 \end{bmatrix}$$

   $$\left[\begin{array}{ccc|c} 4 & 2 & 2 & 14 \\ 2 & 2 & 0 & 4 \\ 2 & 0 & 2 & 10 \end{array}\right] \cong \left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & 5 \end{array}\right] \cong \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -3 \\ 1 & 0 & 1 & 5 \end{array}\right]$$

   So x_3 is free and we get $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} x_3$

   (b) (1 pt) Find the error associated to your least squares solution from part (a)

   $$A\widehat{\mathbf{x}} = \begin{bmatrix} 2 \\ 2 \\ 5 \\ 5 \end{bmatrix}$$

   $$\text{Error} = \|A\widehat{\mathbf{x}} - b\| = \left\| \begin{bmatrix} 1 \\ -1 \\ -3 \\ 3 \end{bmatrix} \right\| = \sqrt{1+1+9+9} = 2\sqrt{5}$$

3. (4 pts) Let $V = \mathbb{P}_2$ and let $\langle p, q \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$. We showed in class that this is an inner product, so you may assume it without proof.

   Find an orthonormal (with resp. to this inner product) basis for V.

   Hint: one (non-orthonormal) basis for $\mathbb{P}_2$ is $\{1, t, t^2\}$

   We start by using Gram-Schmidt to find an orthogonal basis:

- $p_1 = 1$
- $p_2 = t - \frac{\langle t,p_1\rangle}{\langle p_1,p_1\rangle} p_1 = t - \frac{\langle t,1\rangle}{\langle 1,1\rangle} 1 = t - \frac{(-1)\cdot 1+0\cdot 1+1\cdot 1}{1\cdot 1+1\cdot 1+1\cdot 1} 1 = t$
- $p_3 = t^2 - \frac{\langle t^2,p_1\rangle}{\langle p_1,p_1\rangle} p_1 - \frac{\langle t^2,p_2\rangle}{\langle p_2,p_2\rangle} p_2 = t^2 - \frac{\langle t^2,1\rangle}{\langle 1,1\rangle} 1 - \frac{\langle t^2,t\rangle}{\langle t,t\rangle} t = t^2 - \frac{(-1)^2\cdot 1+0^2\cdot 1+1^2\cdot 1}{1+1+1} 1 - \frac{(-1)^2(-1)+0^2 0+1^2 1}{(-1)\cdot(-1)+0\cdot 0+1\cdot 1} t = t^2 - \frac{2}{3}$

So now that we have an orthogonal basis, we just need to make them unit vectors:
$\|1\| = \sqrt{\langle 1,1\rangle} = \sqrt{1+1+1} = \sqrt{3}$
$\|t\| = \sqrt{\langle t,t\rangle} = \sqrt{(-1)\cdot(-1)+0\cdot 0+1\cdot 1} = \sqrt{2}$
$\|t^2 - \frac{2}{3}\| = \sqrt{\langle t^2-\frac{2}{3}, t^2-\frac{2}{3}\rangle} = \sqrt{((-1)^2-\frac{2}{3})((-1)^2-\frac{2}{3}) + (0^2-\frac{2}{3})(0^2-\frac{2}{3}) + (1^2-\frac{2}{3})(1^2-\frac{2}{3})} = \sqrt{\frac{1}{9}+\frac{4}{9}+\frac{1}{9}} = \sqrt{2/3}$
So our othonormal basis is

$$\left\{ \frac{1}{\sqrt{3}}, \frac{t}{\sqrt{2}}, \frac{t^2-\frac{2}{3}}{\sqrt{2/3}} \right\}$$