Name: ______________________

Section: 9-10 11-12 2-3

Math 54 Quiz 8 SOLUTIONS

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You have 20 minutes to complete this quiz. You must show your work.

1. (2 pts)

   (a) Complete the following equation: $\text{Proj}_{\mathbf{u}}\mathbf{y} = \boxed{\frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}}$

   (b) Define what it means for $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ to be an orthonormal set of vectors

   There are many ways to answer this, but the shortest is probably: $\mathbf{v}_i\cdot\mathbf{v}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

2. Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 4 \\ 3 \\ 3 \\ -1 \end{bmatrix}$.

   (a) (2 pts) Show that $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is an orthogonal set

   $$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix} = -1 + 3 + 0 - 2 = 0$$

   $$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = -1 + 0 + 0 + 1 = 0$$

   $$\begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = 1 + 0 + 1 - 2 = 0$$

   (b) (3 pts) Let $W = \text{Span}\,\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. Write $\mathbf{y}$ as $\widehat{\mathbf{y}} + \mathbf{z}$ where $\widehat{\mathbf{y}} \in W, \mathbf{z} \in W^{\perp}$

   Since the $\mathbf{b}'s$ are orthogonal, we can find $\widehat{y}$ by calculating the projection onto each vector:

$$\widehat{\mathbf{y}} = \frac{\begin{bmatrix} 4 \\ 3 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \frac{\begin{bmatrix} 4 \\ 3 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix}}{\begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix} + \frac{\begin{bmatrix} 4 \\ 3 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= 2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \frac{10}{15} \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 4 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{z} = \mathbf{y} - \widehat{\mathbf{y}} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

3. (3 pts) Let $T : \mathbb{P}_2 \to \mathbb{P}_3$ be the linear transformation defined by $T(p) = \int_0^t p(x)dx$. Find the matrix for T with respect to the bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3\}$

(Hint: as an example, $T(3+4t) = \int_0^t (3+4x)dx = 3x + 2x^2|_0^t = 3t + 2t^2 - (0+0) = 3t + 2t^2$)

$$\begin{aligned} T(1) &= \int_0^t 1dx = t \\ T(t) &= \int_0^t xdx = \frac{t^2}{2} \\ T(t^2) &= \int_0^t x^2dx = \frac{t^3}{3} \end{aligned}$$

In $\{1, t, t^2, t^3\}$-coordinates, these are $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{3} \end{bmatrix}$, respectively, so the matrix is:

$$[T] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$