Name: ______________________

Section: 9-10 11-12 2-3

Math 54 Quiz 5 SOLUTIONS

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You have 20 minutes to complete this quiz. You must show your work.

1. (3 pts) Let $\mathcal{B} = \{1+t^2, t+t^2, 1+2t+t^2\}$ and $\mathcal{C} = \{1+2t, 3+t^2, 2-2t+3t^2\}$. You may assume that $\mathcal{B}$ is a basis for $\mathbb{P}_2$

   (a) Show that $\mathcal{C}$ is a basis for $\mathbb{P}_2$

   We have three vectors, namely $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$, in $\mathbb{P}_2$, which has dimension 3 and thus we only need to show they are linearly independent:

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -6 & -6 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

   (b) Suppose $p(t)$ is some polynomial such that $[p]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -6 \\ 5 \end{bmatrix}$. Find $[p]_{\mathcal{C}}$

   If $[p]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -6 \\ 5 \end{bmatrix}$, then $p(t) = 0(1+t^2) - 6(t+t^2) + 5(1+2t+t^2) = 5+4t-t^2$. So then we just need to solve $A\mathbf{x} = \begin{bmatrix} 5 \\ 4 \\ -1 \end{bmatrix}$, where A is the matrix from part (a):

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 2 & 0 & -2 & 4 \\ 0 & 1 & 3 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & -6 & -6 & -6 \\ 0 & 1 & 3 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -2 \end{array}\right]$$

   So we can back substitute to get $x_3 = -1$, so $x_2 = 2$ and $x_1 = 1$. Thus, $[p]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

2. (3 pts) Find a basis for the subspace of $\mathbb{R}^3$ spanned by $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 7 & 1 \\ -1 & -2 & 4 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 6 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   So the pivot columns are 1 and 3 and thus our basis is $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}$

3. (4 pts) Let $T : V \to W$ be a 1-1 linear transformation from the vector space V to the vector space W. Prove that if $\mathbf{v}_1, \mathbf{v}_2, \cdots \mathbf{v}_p$ are vectors in V such that $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \cdots T(\mathbf{v}_p)\}$ are linearly dependent, then $\{\mathbf{v}_1, \mathbf{v}_2, \cdots \mathbf{v}_p\}$ are linearly dependent.

   If $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \cdots T(\mathbf{v}_p)\}$ are linearly dependent, then there are constants $c_1, c_2, \ldots c_p$, not all of which are 0, such that

$$c_1 T(\mathbf{v}_1) + c_2 T(\mathbf{v}_2) + \cdots c_p T(\mathbf{v}_p) = \mathbf{0}_W$$

   Since T is linear, this is the same as

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_p\mathbf{v}_p) = \mathbf{0}_W$$

   Since T is 1-1, we must have

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_p\mathbf{v}_p = \mathbf{0}_V$$

   Since we chose the c_i's such that at least 1 is non-zero, we have shown that $\{\mathbf{v}_1, \mathbf{v}_2, \cdots \mathbf{v}_p\}$ are linearly dependent.