Name: ______________________
Section: 9-10 11-12 2-3

Math 54 Quiz 4 SOLUTIONS

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You have 25 minutes to complete this quiz. You must show your work.

1. (3 pts) Let $A = \begin{bmatrix} 1 & -3 & 2 & -4 \\ -3 & 9 & -1 & 5 \\ 2 & -6 & 4 & -3 \\ -4 & 12 & 2 & 7 \end{bmatrix}$, which is row equivalent to $\begin{bmatrix} 1 & -3 & 2 & -4 \\ 0 & 0 & 5 & -7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

   Find (a) a basis for Nul A, (b) a basis for Col A, and (c) rk A

   (a) We have $x_1 = 3x_2 - 2x_3 + 4x_4, x_2$ is free, $5x_3 = 7x_4, x_4 = 0$, so substituting $x_3 = x_4 = 0$, we get that $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2$, so the basis is just $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

   (b) From the reduced form, we see that columns 1, 3, and 4 are the pivot columns, so the basis is just $\left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ -3 \\ 7 \end{bmatrix} \right\}$

   (c) There are 3 pivot columns, so rk $A = 3$.

2. (3 pts) For any subspaces H and K of a vector space V, we define $H + K$ to be the set of all vectors in V that can be written as the sum of a vector in H and a vector in K. Formally, $H + K = \{\mathbf{v} \in V : \mathbf{v} = \mathbf{h} + \mathbf{k} \text{ for some } \mathbf{h} \in H, \mathbf{k} \in K\}$. Prove that $H + K$ is a subspace of V

   (a) Since H and K are subspaces, we know $\mathbf{0} \in H$ and $\mathbf{0} \in K$. So if we take $\mathbf{h} = \mathbf{k} = \mathbf{0}$ in the definition of $H + K$, we see that $\mathbf{0} + \mathbf{0} \in H + K$ But we know $\mathbf{0} + \mathbf{0} = \mathbf{0}$, so $\mathbf{0} \in H + K$

   (b) Suppose $\mathbf{v} \in H + K$ and $\mathbf{w} \in H + K$. Then by the definition of $H + K$, there are $\mathbf{h_1}, \mathbf{h_2} \in H, \mathbf{k_1}, \mathbf{k_2} \in K$ such that $\mathbf{v} = \mathbf{h_1} + \mathbf{k_1}$ and $\mathbf{w} = \mathbf{h_2} + \mathbf{k_2}$. Then $\mathbf{v} + \mathbf{w} = (\mathbf{h_1} + \mathbf{k_1} + \mathbf{h_2} + \mathbf{k_2} = (\mathbf{h_1} + \mathbf{h_2}) + (\mathbf{k_1} + \mathbf{k_2})$. Since H is a subspace and $\mathbf{h_1}, \mathbf{h_2} \in H$, we know $\mathbf{h_1} + \mathbf{h_2} \in H$. Similarly, $\mathbf{k_1} + \mathbf{k_2} \in K$ and thus $\mathbf{v} + \mathbf{w} \in H + K$ (just take $\mathbf{h} = \mathbf{h_1} + \mathbf{h_2}, \mathbf{k} = \mathbf{k_1} + \mathbf{k_2}$ in the definition of $H + K$

   (c) Suppose $\mathbf{v} \in H + K$ and $c \in \mathbb{R}$. Then $\mathbf{v} = \mathbf{h} + \mathbf{k}$ for some $\mathbf{h} \in H, \mathbf{k} \in K$. So $c\mathbf{v} = c\mathbf{h} + c\mathbf{k}$, and since H, K are subspaces, $c\mathbf{h} \in H$ and $c\mathbf{k} \in K$. So then $c\mathbf{v} \in H + K$ by definition of $H + K$.

   Thus, $H + K$ is a subspace of V

3. (2 pts) Let S be the parallelepiped with edge vectors (1,0,-2), (1, 2, 4), (7,1,0) and let T be the linear transformation with matrix $\begin{bmatrix} 1 & e^2 & \pi \\ 0 & 4 & -\sqrt{3} \\ 0 & 0 & 6 \end{bmatrix}$. Find:

(a) $\text{Vol}(S) = \begin{vmatrix} 1 & 1 & 7 \\ 0 & 2 & 1 \\ -2 & 4 & 0 \end{vmatrix} = 2\begin{vmatrix} 1 & 7 \\ -2 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ -2 & 4 \end{vmatrix}$

(b) $\text{Vol}(T(S)) = |\det T| \cdot \text{Vol}(S) = 24 \cdot 22 = 528$

4. (2 pts) True/False. Determine whether each statement below is true or false. For those that are true, prove it. For those that are false, either provide a counterexample or explain why the statement is false.

   (a) A matrix A has a non-trivial nullspace if and only if its Echelon Form has a row of all zeros

   False. $\begin{bmatrix} 1 & 1 \end{bmatrix}$ is a non-trivial nullspace

   (b) No 4×6 matrix can have a nullspace of dimension 1

   True. The maximum number of pivots is 4, so there are at least 2 nonpivot columns and thus the dimension of the nullspace must be at least 2.

   (c) There is a matrix A with $\det A = 3$ such that some Echelon Form of A has determinant 0.

   False. Since $\det A = 3$, A is invertible, so every matrix that is row equivalent (including Echelon Forms) must be invertible and thus have a non-zero determinant.

   (d) Every subspace of a vector space has infinitely many vectors in it

   False. $\{\mathbf{0}\}$ is a subspace of every vector space and only has 1 vector in it.