10.6 SOLUTIONS

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Note: This is probably filled will all sorts of errors, but the idea/outline is correct for each problem, even if I am missing a few negatives, or make some algebra mistakes.

2. Since this is homogeneous, we know (from section 10.2) that the general solution is
$u(x,t) = \sum_{n=1}^{\infty}(a_n \cos 4nt + b_n \sin 4nt)\sin nx$
Our initial condition $u(x,0) = \sin^2 x$ tells us that we want $\sum_{n=1}^{\infty} a_n \sin nx = \sin^2 x$, so the a_n's should be chosen as the coefficients of the Fourier sine series for $\sin^2 x$:

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} \sin^2 x \sin nx \\ &= \frac{2}{\pi}\int_0^{\pi} \frac{1-\cos 2x}{2}\sin nx \\ &= \frac{1}{\pi}\int_0^{\pi} \sin nx - \frac{1}{\pi}\int_0^{\pi}\cos 2x \sin nx \\ &= 4\frac{(-1)^n - 1}{\pi(n^3-4n)} \end{aligned}$$

Our other initial condition $u_t(x,0) = 1-\cos x$ gives that we need $\sum_{n=1}^{\infty} 4nb_n \sin nx = 1-\cos x$, so we compute

$$b_n = \frac{2}{4n\pi}\int_0^{\pi}(1-\cos x)\sin nx = \frac{-1+(-1)^n(1-2n^2)}{2n^2(n^2-1)\pi}$$

So $u(x,t) = \displaystyle\sum_{n=1}^{\infty}\left(4\frac{(-1)^n-1}{\pi(n^3-4n)}\cos 4nt + \frac{-1+(-1)^n(1-2n^2)}{2n^2(n^2-1)\pi}\sin 4nt\right)\sin nx$

4. General solution is $u(x,t) = \displaystyle\sum_{n=1}^{\infty}(a_n \cos 3nt + b_n \sin 3nt)\sin nx$.
We need $\sum_{n=1}^{\infty} a_n \sin nx = \sin 4x + 7\sin 5x$, so $a_4 = 1, a_5 = 7, a_n = 0$ for all other n.
We also need $\displaystyle\sum_{n=1}^{\infty} 3nb_n \sin nx = \begin{cases} x & 0 < x < \pi/2 \\ \pi - x & \pi/2 < x < \pi \end{cases}$
Then

$$3nb_n = \frac{2}{\pi}\int_0^{\pi/2} x\sin nx + \frac{2}{\pi}\int_{\pi/2}^{\pi}(\pi - x)\sin nx = \frac{4\sin\frac{n\pi}{2}}{n^2\pi}$$

Now just plug these into our original general solution to get:

$$u(x,t) = \cos 12t \sin 4x + 7\cos 5t \sin 5x + \sum_{n=1}^{\infty}\frac{4\sin\frac{n\pi}{2}}{3n^3\pi}\sin 3nt \sin nx$$

8. We guess that $u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin nx$.
The fourier sine coefficients of $x\sin t$ with $T = \pi$ are given by $a_n = \frac{2\sin t}{\pi}\int_0^{\pi} x\sin nx = \frac{2(-1)^n \sin t}{n}$.
Plugging our guess and the fourier sine series of $x\sin t$ into the original differential equation gives:

$$\sum_{n=1}^{\infty} u_n''(t)\sin nx = -\sum_{n=1}^{\infty} n^2 \sin nx + \sum_{n=1}^{\infty}\frac{2(-1)^n \sin t}{n}\sin nx$$

Equating these term-by-term and dividing through by the $\sin nx$, we see that we need:

$$u_n'' + n^2 u_n = \frac{2(-1)^n \sin t}{n}$$

Then $(u_n)_h = C_1 \cos nt + C_2 \sin nt$

If $n \neq 1$, we guess $(u_n)_p = A \sin t + B \cos t$, so $(u_n)_p'' = -A \sin t - B \cos t$.

Then we need $(n^2-1)A \sin t + (n^2-1)\cos t = \frac{2(-1)^n \sin t}{n}$, which tells us that $B = 0$ and $A = \frac{2(-1)^{n+1}}{n(n^2-1)}$

If $n = 1$, we guess $(u_1)_p = t(A \sin t + B \cos t)$, so $(u_1)_p'' = 2A \cos t - 2B \sin t - t(A \sin t + B \cos t)$.

Plugging into our equation (with $n = 1$) gives that we need $2A \cos t - 2B \sin t = -2 \sin t$, so $A = 0, B = 1$, so $(u_1)_p = t \cos t$

Combining all this gives

$$u(x,t) = (a_1 \cos t + b_1 \sin t + t \cos t) \sin x + \sum_{n=2}^{\infty} \left(a_n \cos nt + b_n \sin nt + \frac{2(-1)^{n+1}}{n(n^2-1)} \sin t \right) \sin nx$$

We want $u(x,0) = 0$, which means we need $a_1 \sin x + \sum_{n=2}^{\infty} a_n \sin nx = 0$, so all the a_i's should be 0.

We also want $u_t(x,0) = 0$. So after plugging in that $a_n = 0$ for all n, we get:

$$\begin{aligned} u_t(x,t) &= (b_1 \cos t + \cos t - t \sin t) \sin x + \sum_{n=2}^{\infty} \left(n b_n \cos nt + \frac{2(-1)^{n+1}}{n(n^2-1)} \cos t \right) \sin nx \\ u_t(x,0) &= b_1 + 1 + \sum_{n=2}^{\infty} (n b_n + \frac{2(-1)^n}{n(n^2-1)}) \sin nx \end{aligned}$$

To make this be 0, we want $b_1 + 1 = 0$ and $n b_n + \frac{2(-1)^n}{n(n^2-1)} = 0$. This happens when $b_1 = -1$ and $b_n = \frac{2(-1)^n}{n^2(n^2-1)}$

Substituting this all back in to our general solution gives

$$u(x,t) = (-\sin t + t \cos t) \sin x + \sum_{n=2}^{\infty} \left(\frac{2(-1)^n}{n^2(n^2-1)} + \frac{2(-1)^{n+1}}{n(n^2-1)} \right) \sin nt \sin nx$$

10. Because all the "'bad"' things happen at x values, we'll guess that $u(x,t) = w(x,t) + v(x)$ where w is a solution to
$w_{tt} = \alpha^2 w_{xx}; w(0,t) = w(L,t) = 0$. Then $u_{xx} = w_{xx} + v''(x)$ and $u_{tt} = w_{tt}$, so we know that we need

$$w_t t = \alpha^2 (w_{xx} + v''(x))$$

Because of our asumptions about w, this reduces to $\alpha v''(x) = 0$, so we know $v(x) = Cx + D$.

Plugging $w + v$ into the given boundary conditions gives $w(0,t) + v(0) = U_1, w(L,t) + v(L) = U_2$ Again, we use what we are assuming about w to conclude that we must have $v(0) = U_1, v(L) = U_2$.

That is, we need $U_1 = D$ and $U_2 = CL + D$, so $D = U_1$ and $C = \frac{U_2 - U_1}{L}$. So we have

$$v(x) = \frac{U_2 - U_1}{L} x + U_1$$

Now let's find w. We want w to satisfy:

$$\begin{aligned} w_{tt} &= \alpha^2 w_{xx} \\ w(0,t) &= 0 \\ w(L,t) &= 0 \\ w(x,0) + v(x) &= f(x) \\ w_t(x,0) &= g(x) \text{ (since } v_t(x) = 0) \end{aligned}$$

This is just a standard wave equation that we know has general solution

$$w(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi\alpha}{L} t + b_n \sin \frac{n\pi\alpha}{L} t \right) \sin \frac{n\pi x}{L}$$

We want $w(x,0) = f(x) - (\frac{U_2 - U_1}{L} x + U_1)$, which means we want

$$\sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} = f(x) - (\frac{U_2 - U_1}{L} x + U_1)$$

So $a_n = \dfrac{2}{L} \displaystyle\int_0^L (f(x) - (\frac{U_2 - U_1}{L} x + U_1)) \sin \frac{n\pi x}{L} dx$

We also want $w_t(x,0) = g(x)$, which means we want

$$\sum_{n=1}^{\infty} \frac{n\pi\alpha}{L} b_n \sin \frac{n\pi x}{L} = g(x)$$

So $b_n = \dfrac{L}{n\pi\alpha} \dfrac{2}{L} \displaystyle\int_0^L g(x) \sin \frac{n\pi x}{L}$

So in summary, we have

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi\alpha}{L} t + b_n \sin \frac{n\pi\alpha}{L} t \right) \sin \frac{n\pi x}{L} + \frac{U_2 - U_1}{L} x + U_1$$

where a_n and b_n are given above.