

Math 54 ODE review¹

1. Separable Equations

- **When to use:** When you can write it as $y' = f(y)g(x)$.
- **How to solve:** Rewrite as $\frac{dy}{f(y)} = g(x)dx$ and integrate both sides.
- **Reminders, Tips, and Tricks:**
 - Separable things are good, so always check this first. If there are things like y^2 floating around, it's a pretty good hint its probably separable (since it's not linear).
 - Don't forget about the constant solutions you get from solving $f(y) = 0$
 - Remember your +C's!

2. Linear (First-Order) DiffEqs

- **When to use:** When it's not separable and y'' doesn't appear anywhere.
- **How to solve:** Rewrite equation in the form $y' + P(x)y = Q(x)$ and compute $I = e^{\int P(x)dx}$. Then $y = \frac{\int IQdx}{I}$
- **Reminders, Tips, and Tricks:**
 - Linear means that y and y' are never squared, cubed, square-rooted, etc.
 - Remember your +C's!

3. Linear Second-Order Homogeneous DiffEqs

- **When to use:** Whenever y'' shows up.
- **How to solve:** Write as $ay'' + by' + cy = 0$ (a, b, c must be constants) and solve the equation $ar^2 + br + c = 0$.
- **Reminders, Tips, and Tricks:**
 - Two Real Roots ($b^2 - 4ac > 0$): $C_1e^{r_1x} + C_2e^{r_2x}$
 - Repeated Root ($b^2 - 4ac = 0$): $C_1e^{rx} + C_2xe^{rx}$
 - Complex Roots ($b^2 - 4ac < 0$, ie $r = \alpha \pm \beta i$): $e^{\alpha x}(C_1 \cos(\beta x) + C_2 \sin(\beta x))$
 - This only works when $a, b,$ and c are constants!

4. Linear Second-Order Non-homogeneous DiffEqs

- **When to use:** Whenever it's $ay'' + by' + cy = G(x)$, with $G(x) \neq 0$
- **How to solve:** Solve the homogeneous part (see above) and then find a particular solution. Then $y = y_p + y_h$

– **Variation of parameters** Call the 2 linearly independent solutions y_1 and y_2 and then calculate

$$u_1 = -\int \frac{y_2 G(x)}{y_1 y_2' - y_1' y_2}, u_2 = \int \frac{y_1 G(x)}{y_1 y_2' - y_1' y_2}$$

Then $y_p = u_1 y_1 + u_2 y_2$

– **Undetermined coefficients**

Take a "guess" at the form of y_p and then solve for the specifics. Some good first guesses:

$G(x)$	Guess for y_p
$\cos(\alpha x)$ or $\sin(\alpha x)$	$A \cos(\alpha x) + B \sin(\alpha x)$
$a_0 + a_1 x + \dots + a_n x^n$	$A_0 + A_1 x + \dots + A_n x^n$
e^{rx}	Ae^{rx}

Plug this guess into the differential equation and find coefficients that give you $G(x)$. If your first guess fails, multiply by x and try again. Rinse and repeat.

¹We will start with the second order stuff in this course, but it might be helpful to at least briefly review it beforehand

Examples

5. $y' = y^2 \cos x$

This is separable, so write it as

$$\begin{aligned}\frac{dy}{y^2} &= \cos x dx \\ \int \frac{dy}{y^2} &= \int \cos x dx \\ -\frac{1}{y} &= \sin x + C \\ y &= -\frac{1}{\sin x + C}\end{aligned}$$

6. $xy' - 2y = x^2$

Divide through by x to get $y' - \frac{2}{x}y = x$, so $P(x) = -2/x$, $Q(x) = x$ Then we have:

$$\begin{aligned}I &= e^{\int -\frac{2}{x}} = e^{-2\ln(x)} = x^{-2} \\ y &= \frac{1}{I} \int IQ \\ &= x^2 \int \frac{1}{x} \\ &= x^2(\ln x + C) \\ &= x^2 \ln(x) + Cx^2\end{aligned}$$

7. $y'' - y = e^x$

The characteristic equation is $r^2 - 1 = 0$, which has solutions $r = 1, -1$. So $y_h = C_1e^x + C_2e^{-x}$

Our guess for a particular solution is $y_p = Ae^x$, so $y'_p = Ae^x$ and $y''_p = Ae^x$. Substituting in, we get $y''_p - y_p = 0$.

We multiply our old guess by x and try again with $y_p = Axe^x$: $y'_p = Ae^x + Axe^x$, $y''_p = 2Ae^x + Axe^x$, so $y''_p - y_p = 2Ae^x$. To make it equal to $G(x) = e^x$, we take $A = \frac{1}{2}$ and get $y = C_1e^x + C_2e^{-x} + \frac{1}{2}xe^x$