December 6^{th} WS Solutions

1\. a) $\int \frac{\sin x}{\cos^{101} x}\,dx$

$u = \cos x$

$du = -\sin x\,dx$

$-\int \frac{1}{u^{101}}\,du = -\int u^{-101}\,du = -\frac{u^{-100}}{-100} + C = \frac{1}{\cos^{100} x \, 100} + C = \frac{1}{100}\sec^{100} x + C$

b) $\int \frac{1}{x\sqrt{x-1}}\,dx$

$u = \sqrt{x-1} \Rightarrow u^2 + 1 = x$

$du = \frac{1}{2\sqrt{x-1}}\,dx \Rightarrow 2du = \frac{1}{\sqrt{x-1}}\,dx$

$2\int \frac{1}{u^2+1}\,du = 2\tan^{-1}(u) + C = 2\tan^{-1}(\sqrt{x-1}) + C$

c) Start with long division:

$$\begin{array}{r} 1 \\ x^3+1 \overline{) x^3 + 0x^2 + 2x + 0} \\ -x^3 \qquad\qquad +1 \\ \hline 2x - 1 \end{array}$$

$\frac{x^3 + 2x}{x^3+1} = 1 + \frac{2x-1}{x^3+1}$

$\frac{2x-1}{x^3+1} = \frac{2x-1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$

$A(x^2-x+1) + (Bx+C)(x+1) = 2x-1$

$Ax^2 - Ax + A + Bx^2 + Cx + Bx + C = 2x - 1$

$(A+B)x^2 + (-A+B+C)x + A + C = 2x - 1$

$$\left.\begin{array}{l} A+B=0 \\ -A+B+C=2 \\ A \quad\;\; +C=-1 \end{array}\right\} \begin{array}{l} 2B+C=2 \\ B+2C=-1 \\ \Leftrightarrow 2B+4C=-2 \end{array} \Bigg\} -3C=4 \Rightarrow \boxed{C=-\frac{4}{3}}$$

$$B-\frac{8}{3}=-1$$

$$\boxed{\begin{array}{l} B=\frac{5}{3} \\ A=-\frac{5}{3} \end{array}}$$

$$\frac{x^3+2x}{x^3+1} = 1+\frac{-5/3}{x+1}+\frac{5/3\,x-4/3}{x^2-x+1}$$

d) Find the length of $y=\frac{1}{x^2}$ b/w $x=1$ and $x=2$:

$$L=\int_1^2\sqrt{1+\left(\left(\frac{1}{x^2}\right)'\right)^2}=\int_1^2\sqrt{1+\left(-\frac{2}{x^3}\right)^2}=\int_1^2\sqrt{1+\frac{4}{x^6}}$$

$$=\int_1^2\frac{\sqrt{x^6+4}}{x^3} = \text{yuk...}$$ this problem probably shouldn't have been here

e) $\int_0^2 \frac{dx}{4x-5}$ is improper b/c $\frac{1}{4x-5}$ is not defined at $x=\frac{5}{4}$

$$\int_0^2\frac{dx}{4x-5}=\underbrace{\lim_{t\to\frac{5}{4}^-}\int_0^t\frac{dx}{4x-5}}_{u=4x-5\Rightarrow du=4dx}+\underbrace{\lim_{s\to\frac{5}{4}^+}\int_s^2\frac{dx}{4x-5}}_{\text{same sub here if you chose this one}}$$

$$=\lim_{t\to0^-}\frac{1}{4}\int_{-5}^t\frac{1}{u}\,du$$

$$=\frac{1}{4}\int_{-5}^0\frac{1}{u}\,du,$$ which we know is a divergent p-integral

2. a) i) $\sum_{n=0}^{\infty} \frac{1000^n}{n!}$

Ratio test: $\lim_{n\to\infty} \left| \frac{1000^{n+1}}{(n+1)!} \frac{n!}{1000^n} \right| = \lim \left| \frac{1000}{n+1} \right| = 0 < 1$

$\Rightarrow$ abs. conv

ii) $\sum_{n=10}^{\infty} \frac{1}{n(\ln n)^{3/2}}$ This is pos and dec, so use integral test.

$\int_{10}^{\infty} \frac{1}{x(\ln x)^{3/2}} dx = \lim_{t\to\infty} \int_{10}^{t} \frac{1}{x(\ln x)^{3/2}} dx$

$u = \ln x$, $du = \frac{1}{x} dx$

$= \lim_{t\to\infty} \int_{\ln 10}^{t} \frac{1}{u^{3/2}} du = \int_{\ln 10}^{\infty} \frac{1}{u^{3/2}} du$, which is a conv. p-integral

$\therefore$ By the integral test, $\sum_{n=10}^{\infty} \frac{1}{n(\ln n)^{3/2}}$ converges

iii) $\sum_{n=1}^{\infty} \sin\left(\pi \frac{n+1}{n}\right) - \sin\left(\pi \frac{n+2}{n+1}\right)$

Telescoping:

$S_n = \sin\left(\pi \frac{2}{1}\right) - \cancel{\sin\left(\pi \frac{3}{2}\right) + \sin\left(\pi \frac{3}{2}\right)} - \cancel{\sin\left(\pi \frac{4}{3}\right)} + \dots \cancel{\sin\left(\pi \frac{n+1}{n}\right)} - \sin\left(\pi \frac{n+2}{n+1}\right)$

$= \sin(2\pi) - \sin\left(\pi \frac{n+2}{n+1}\right) = -\sin\left(\pi \frac{n+2}{n+1}\right)$

Then $\lim_{n\to\infty} S_n = -\sin(\pi \cdot 1) = -\sin(\pi) = 0$

So $\sum_{n=1}^{\infty} \sin\left(\pi \frac{n+1}{n}\right) - \sin\left(\pi \frac{n+2}{n+1}\right)$ conv. and $= 0$

b) i) $\ln(1+x) = \int \frac{1}{1+x} = \int \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C$

plugging in $x=0$ gives

$$\underbrace{\ln(1)}_{=0} = \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{0^{n+1}}{n+1}}_{=0} + C, \text{ so } C = 0$$

and $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$

ii) $\ln(1\cancel{+}x) = \ln(1+(-x)) = \sum_{n=0}^{\infty} (-1)^n \frac{(-x)^{n+1}}{n+1}$

$$= \sum_{n=0}^{\infty} (-1)^n (-1)^{n+1} \frac{x^{n+1}}{n+1}$$

$$= \sum_{n=0}^{\infty} (-1) \frac{x^{n+1}}{n+1}$$

$$= -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

iii) $\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-y)$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

$$= \sum_{n=0}^{\infty} \underbrace{\left((-1)^n + 1\right)}_{\substack{=0 \text{ if } n \text{ is odd} \\ =2 \text{ if } n \text{ is even}}} \frac{x^{n+1}}{n+1}$$

$$= \sum_{n=0}^{\infty} 2 \frac{x^{2n+1}}{2n+1}$$

$$= 2 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$$

c) i) $\frac{n^2-n+2}{\sqrt[4]{n^{10}+n^5+3}} \sim \frac{n^2}{\sqrt[4]{n^{10}}} = \frac{n^2}{n^{10/4}} = \frac{1}{n^{10/4-2}} = \frac{1}{n^{1/2}}$, diverges.

ii) $\frac{1+e^{-n}}{n}$ is decreasing and $\to 0$, so $\sum (-1)^n \frac{1+e^{-n}}{n}$ conv. by alt. series test

However, $\frac{1+e^{-n}}{n} > \frac{1}{n}$, so $\sum \frac{1+e^{-n}}{n}$ div by comp test and thus $\sum (-1)^n \frac{1+e^{-n}}{n}$ is cond. conv.

d) $\sum_{n=2}^{\infty} (-1)^n \frac{(4x+1)^n}{n}$

Ratio test: $\lim_{n\to\infty} \left| \frac{(4x+1)^{n+1}}{n+1} \cdot \frac{n}{(4x+1)^n} \right| = |4x+1|$

So we need

$$|4x+1| < 1$$
$$4\left|x+\tfrac{1}{4}\right| < 1$$
$$\left|x+\tfrac{1}{4}\right| < \tfrac{1}{4}$$

$R = \frac{1}{4}$ and $a = -\frac{1}{4}$

$I = \left(-\frac{1}{2}, 0\right]$

$x = -\frac{1}{2}$:
$\sum (-1)^n \frac{(-1)^n}{n} = \sum \frac{1}{n}$
div.

$x = 0$:
$\sum (-1)^n \frac{1^n}{n} = \sum \frac{(-1)^n}{n}$
conv. by alt. series test

3. a) $y' = \cos^2 y \ln x$ This is separable, so:

$$\frac{dy}{\cos^2 y} = \ln x \, dx$$

$$\int \sec^2 y \, dy = \int \ln x \, dx$$

$$\tan y = \cancel{x \ln x} \; x\ln x - x + C$$

$$y = \tan^{-1}(x\ln x - x + C)$$

b) $y' + \cos x \; y = \sin x \cos x$ is linear:

$$I = e^{\int \cos x} = e^{+\sin x}$$

$$\int IQ = \int e^{\sin x} \sin x \cos x \, dx$$

$u = \sin x$

$du = \cos x$

$$= \int u e^u \, du$$

$u = u$ $\quad dv = e^u du$

$du = du$ $\quad v = e^u$

$$= ue^u - \int e^u$$

$$= ue^u - e^u + C$$

$$= \sin x \, e^{\sin x} - e^{\sin x} + C$$

$$y = \frac{\int IQ}{I} = \sin x - 1 + \frac{C}{e^{\sin x}}$$

c) $x^2y' = y^2 + 3yx + x^2$ is the same as $y' = \left(\frac{y}{x}\right)^2 + 3\left(\frac{y}{x}\right) + 1$

so we do $v = \frac{y}{x}$.

Then $y = vx$, so $y' = v'x + v$ and we have

$$v'x + v = v^2 + 3v + 1$$

$$v' = \frac{v^2+2v+1}{x} = \frac{(v+1)^2}{x}$$

$$\frac{dv}{(v+1)^2} = \frac{1}{x}dx$$

$$-\frac{1}{v+1} = \ln x + C$$

$$v+1 = -\frac{1}{\ln x + C}$$

$$v = -\frac{1}{\ln x + C} - 1$$

$$y = xv = -\frac{x}{\ln x + C} - x$$

d) $y'\cos x = y\sin x + e^x\cos x$, so

$$y'\cos x - y\sin x = e^x\cos x$$

$$y' - y\frac{\sin x}{\cos x} = e^x$$

$$I = e^{-\int \tan x} = e^{+\ln(\cos x)} = \cos x$$