Math 1B Discussion Section SOLUTIONS

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1. (a) Show that $xy' = y$ is separable, homogeneous, and linear, by putting it into the proper form for each technique.
   Linear: $y' - \frac{y}{x} = 0$
   Separable: $\frac{dy}{y} = \frac{dx}{x}$
   Homogeneous: $y' = \frac{y}{x}$

   (b) Now solve the equation using the separation of variables technique and then again using an Integrating Factor. Do you get the same thing? Do they both satisfy the original DE?
   Separable:

$$\begin{aligned} \frac{dy}{y} &= \frac{dx}{x} \\ \ln(y) &= \ln(x) + C \\ y &= e^{\ln(x)+C} \\ y &= Cx \end{aligned}$$

   Linear: $y' - \frac{y}{x} = 0$, so $P = -1/x$, $Q = 0$

$$\begin{aligned} I &= e^{\int -\frac{1}{x}dx} \\ &= e^{-\ln x} \\ &= \frac{1}{x} \\ y &= \frac{\int \frac{1}{x}0}{1/x} \\ &= \frac{C}{1/x} = Cx \end{aligned}$$

2. For each of the following equations, decide whether it is separable, homogeneous, linear, or none of these. For those that are linear, find the general solution

   (a) $yy' = x\sqrt{1+x^2}\sqrt{1+y^2}$
   Separable

   (b) $xy' - 2y = x^3$

Linear:

$$\begin{aligned}
xy' - 2y &= x^3 \\
y' - \frac{2}{x}y &= x^2 \text{ (so } P = -\frac{2}{x}, Q = x^2) \\
I &= e^{\int -\frac{2}{x}} \\
&= e^{-2\ln(x)} = x^{-2} \\
y &= \frac{1}{x^{-2}} \int x^{-2}x^2 \\
&= x^2 \int 1 \\
&= x^2(x + C) \\
&= x^3 + Cx^2
\end{aligned}$$

(c) $xy' = y + x\cos^2(y/x)$
Homogeneous: $y' = \frac{y}{x} + \cos^2(y/x)$

(d) $y' = 2 + 2x^2 + y + x^2y$
Separable: $y' = 2 + 2x^2 + y + x^2y = 2(1 + x^2) + y(1 + x^2) = (2 + y)(1 + x^2)$
Linear: (note: this is probably not the right way to do this problem–separable is definitely the correct choice here)

$$\begin{aligned}
y' &= 2(1 + x^2) + y(1 + x^2) \\
y' - (1 + x^2)y &= 2(1 + x^2) \text{ (so } P(x) = -(1 + x^2) \text{ and } Q(x) = 2(1 + x^2)) \\
I &= e^{\int -(1+x^2)} = e^{-x-x^3/3} \\
y &= \frac{1}{e^{-(x+x^3/3)}} 2 \int (1 + x^2)e^{-(x+x^3/3)} \\
&= -2\frac{e^{-(x+x^3/3)} + C}{e^{-(x+x^3/3)}} \\
&= -2 + Ce^{x+x^3/3}
\end{aligned}$$

(e) $y' = x + y$
Linear: $P(x) = -1, Q(x) = x$

(f) $1 + 2xy^2 + 2x^2yy' = 0$
None of the above

3. Last week, we found a model for the amount of salt in a tank that started with 1000L of pure water, had 5L/min of 1kg/L saltwater entering it, and had 6L/min of solution leaving to be

$$S' = 5 - \frac{6S}{1000 - t}$$

Solve this linear ODE to find an equation for the amount of salt in the tank at any given time $t < 1000$
$P(x) = \frac{6}{1000-t}, Q(x) = 5$. Solve as above.

4. For an object of mass m falling towards the ground, the force of air resistance is proportional to the object's speed (ie, $R = kv$) and the force of gravity is a constant mg. Furthermore, Newton's Second Law of motion says that Total Force $= ma$, where a is the acceleration.

(a) Let $y(t)$ be a function for the objects height at time t and use the above information to come up with a differential equation that models a free falling object.
First of all, $v = y'$ and $a = y''$. Then we know $F = F_{down} - F_{up} = mg - vk = mg - ky'$. Then from $F = ma$, we have $my'' = mg - ky'$

(b) Re-write your equation in the form of a first-order linear differential equation based on velocity.
If we make ths substitution $v = y'$, we get $mv' = mg - kv$

(c) Solve your equation to find velocity as a function of time

$mv' + kv = mg$, so $P(t) = k/m$ and $Q(t) = g$

Then $I = e^{\int k/m} = e^{kt/m}$, so $v = \frac{1}{e^{kt/m}} \int e^{kt/m}(g)$

Simplifying, we get

$$v = e^{-kt/m}\left(\frac{gm}{k}e^{kt/m} + C\right) = \frac{gm}{k} + Ce^{-kt/m}$$

(d) Compute $\lim_{t\to\infty} v(t)$

$\frac{gm}{k}$

(e) Your answer to part (d) is called the terminal velocity of the object. Explain why the object will reach the same terminal velocity regardless of whether you simply drop the object or launch it downwards.

If you throw it downward at faster than terminal velocity, the differential equation we wrote for v shows that the velocity will actually decrease until it returns to $\frac{gm}{k}$

5. Go back to problem 2 and solve the rest of the differential equations.