Math 1B Discussion Section SOLUTIONS

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1. What is the difference between a Taylor Series and a Maclaurin Series?

   A Maclaurin series is just a Taylor series with $a = 0$

2. Find a Taylor Series expansion for each of the following functions and find the associated radius of convergence.

   (a) $f(x) = \sin x$, centered at $x = \pi/2$

n	$f^{(n)}(x)$	$f^{(n)}(\pi/2)$
0	$\sin x$	1
1	$\cos x$	0
2	$-\sin x$	-1
3	$-\cos x$	0
4	$\sin x$	1
5	$\cos x$	0
6	$-\sin x$	-1

   So the Taylor series is $1 - \frac{1}{2!}(x-\pi/2)^2 + \frac{1}{4!}(x-\pi/2)^4 - \frac{1}{6!}(x-\pi/2)^6 + \cdots$

   This can be written as $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(x-\pi/2)^{2n}$

   (b) $\sin^2(x)$, centered at 0. Hint: $2\sin x \cos x = \sin(2x)$

   Similar to above.

   (c) $\frac{1}{\sqrt{x}}$, centered at 4.

n	$f^{(n)}(x)$	$f^{(n)}(4)$
0	$x^{-1/2}$	$\frac{1}{2}$
1	$\frac{-1}{2}x^{-3/2}$	$-\frac{1}{2^4}$
2	$\frac{-1}{2}\frac{-3}{2}x^{-5/2}$	$\frac{3}{4}\frac{1}{2^5} = \frac{3}{2^7}$
3	$\frac{3}{4}\frac{-5}{2}x^{-7/2}$	$\frac{-3\cdot5}{8}\frac{1}{2^7} = -\frac{3\cdot5}{2^{10}}$
4	$\frac{-3\cdot5}{8}\frac{-7}{2}x^{-9/2}$	$\frac{3\cdot5\cdot7}{2^{13}}$

   Pugging this into the Taylor Series formula, we get $\sum_{n=0}^{\infty} \frac{(-1)^n(2n-1)!!}{2^{3n+1}n!}x^n$

   (d) $\frac{1}{\sqrt[4]{16+x}}$ (Hint: don't try to take derivatives)

   This is the same as $(16+x)^{-1/4} = (16(1+\frac{x}{16}))^{-1/4} = \frac{1}{2}(1+\frac{x}{16})^{-1/4} = \frac{1}{2}\sum_{n=0}^{\infty}\binom{-1/4}{n}(\frac{x}{16})^n$

To simplify this, we note that

$$\begin{aligned}\binom{-1/4}{n} &= \frac{(-1/4)(-1/4-1)(-1/4-2)\cdots(-1/4-(n-1))}{n!} \\ &= \frac{\frac{-1}{4}\frac{-5}{4}\frac{-9}{4}\cdots\frac{-1-4(n-1)}{4}}{n!} \\ &= (-1)^n\frac{1\cdot 5\cdot 9\cdots(4n-3)}{4^n n!}\end{aligned}$$

So if we make the (semi-reasonable) convention that $a!!!!$ is the product of every 4th term below a, then the Taylor series is $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(4n-3)!!!!}{4^n n!}x^n$

3. We already know that $\sin x = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$. Using this, find a Maclaurin series for:

   (a) $\sin(x^3)$
   $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{6n+3}}{(2n+1)!}$

   (b) $x\sin(x^3)$
   $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{6n+4}}{(2n+1)!}$

   (c) Can you use similar techniques to find a Taylor Series for $\sin(\sqrt{x+1})$? How about for $\cos(\sqrt{x+1})$?
   $\sin(\sqrt{x+1}) = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\sqrt{x+1}^{2n+1}}{(2n+1)!}$, which is NOT a power series since it includes $\sqrt{x+1}$
   $\cos(\sqrt{x+1}) = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\sqrt{x+1}^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n(x+1)^n}{(2n)!}$ and this IS power series

4. (All about 0!)

   (a) How can you find 5! if you already know 4!?
   $5! = 5\cdot 4!$

   (b) How about 2! if you know 1!?
   $2! = 2\cdot 1!$

   (c) Why do the above two examples give a good hint as to why we should say $0! = 1$
   We would expect then that $1! = 1\cdot 0!$, so we need $0! = 1$

5. (More proof that infinity is very, very strange)

   (a) What is $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$
   This is a geometric series with $a=1, r=1/2$, so its sum is $\frac{1}{1-1/2}=2$

   (b) Find a power series for $\dfrac{x}{(1-x)^2}$
   $\dfrac{x}{(1-x)^2} = x\dfrac{d}{dx}\dfrac{1}{1-x} = x\displaystyle\sum_{n=0}^{\infty}nx^{n-1} = \sum_{n=0}^{\infty}nx^n$

   (c) Use your answer to (b) to find $\displaystyle\sum_{n=0}^{\infty}n\left(\frac{1}{2}\right)^n$
   This is just part (b) with $x=1/2$ plugged in, so we can plug into the formula $\frac{x}{(1-x)^2}$ and get $\frac{1/2}{(1-1/2)^2}=2$, which amazingly enough is the same as (a).

6. Let $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$.

(a) Sketch a graph of this function.

You can look this up online if you really care.

(b) Find $f'(0)$ by using the definition of derivative.

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{e^{-1/x^2}}{x} = 0$$

(c) It can be shown (though it's rather difficult) that $f^{(n)}(0)$ exists and is 0 for all n. What does this mean for the Taylor series of f at 0? What is its radius of convergence?

This means the Taylor series is just $\sum 0 = 0$

(d) For what values of x does $f(x)$ equal this Taylor Series?

Only at $x = 0$.