Math 1B Discussion Section Problems

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You should work on the following problems in groups of 3 or 4. Try to get through as many as you can, but you aren't expected to finish everything. Instead, you should make sure everyone in your group knows **how** to solve all the problems and not just the answers.

1. Starting from the geometric series, find a power series representation for each of the following functions, and determine the radius of convergence. DO NOT do a Taylor expansion.

   (a) $\dfrac{1}{1+x^3}$
   Substitute $t = -x^3$ to get $\sum_{n=0}^{\infty}(-x^3)^n$. The radius is just 1.

   (b) $\dfrac{1}{4+x^2}$
   We re-write this as $\frac{1}{4}\frac{1}{1-(-x^2/4)}$, which we know is just $\frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{x^2}{4}\right)^n$. This converges when $|-x^2/4| < 1$, ie when $|x| < 2$

   (c) $\arctan(x)$
   $\arctan(x) = \int \frac{1}{1+x^2} = \int \sum_{n=0}^{\infty}(-x^2)^n = \int \sum (-1)^n x^{2n} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} + C$
   To solve for the constant, we just plug in $x = 0$ to get $arctan(0) = (\sum 0) + C$, so $C = 0$.
   For the radius, we go back to where we turned it into a series and get $|-x^2| < 1$, ie $|x| < 1$, so integrating didn't change the radius and thus it's still 1. If we wanted to find the actual interval, we'd have to plug in 1 and -1.

   (d) $\dfrac{\ln(1+x)}{x}$
   We know $\ln(1+x) = \sum_{n=1}^{\infty}\frac{(-x)^n}{n}$, so we just divide by x and get $\ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^n x^{n-1}}{n}$

   (e) $\dfrac{1}{(1-x)^2}$
   $\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}\sum_{n=0}^{\infty}x^n = \sum_{n=1}^{\infty}nx^{n-1}$.
   This has radius of convergence 1, since that was the radius of the geometric series we substituted for $\frac{1}{1-x}$ and all we did was take a derivative.

2. What function is represented by each of the following power series?

   (a) $\displaystyle\sum_{n=0}^{\infty}x^n$
   This is just the geometric series, so it is $\frac{1}{1-x}$ whenever $|x| < 1$

   (b) $\displaystyle\sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}x^n$
   This is just 1/2 of the second derivative of $\sum x^n$, so it's $\frac{1}{2}\left(\frac{1}{1-x}\right)'' = \frac{1}{(1-x)^3}$

(c) $\sum_{n=1}^{\infty} nx^{2n-1}$ [Hint: what if it were $2nx^{2n-1}$ inside the sum?]

This is just 1/2 of the derivative of $\sum x^{2n}$, which is itself a power series for $\frac{1}{1-x^2}$, so this series $\frac{1}{2}\left(\frac{1}{1-x^2}\right)' = \frac{x}{(1-x^2)^2}$

3. When we say $f(x) = \sum_{n=0}^{\infty} a_n x^n$, what we really mean is that these two things give the same value no matter what you plug in for x (as long as it's in the interval of convergence).

(a) In particular, when we plug in $x = 0$, both sides should give the same result. Based on this, what must a_0 be?

$f(x) = \sum a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$, so if we plug in 0, we get the equation $f(0) = a_0 + a_1 0 + a_2 0^2 + \cdots = a_0$ and thus $a_0 = f(0)$

(b) If all the values of the power series and the function are the same, then all derivatives should be the same too (to see this, think about the formal definition of derivative). Using this, how can you determine what a_1 must be? a_2?

Let's take the derivative to get:

$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$

As before, plug in 0 and get $f'(0) = a_1$

Similarly,

$f''(x) = 2a_2 + 6a_3 x + \cdots$, so plugging in 0 gives $a_2 = f''(0)/2$

4. Show that if $\sum a_n x^n$ has infinitely many coefficients that are non-zero integers, then the radius of convergence is at most 1. Note: this is a hard problem to prove formally–if you're getting stuck with the symbols, try to figure out intuitively why this must be true.

Hint1: power series are absolutely convergent on any closed interval contained in their interval of convergence.

Hint2: It may be helpful to prove that any sub-series of an absolutely convergent series also converges.

If the radius of convergence is bigger than 1, then $\sum a_n x^n$ converges for $x = 1$ and thus $\sum a_n$ converges. Therefore we must have $\lim a_n = 0$, which is not possible if infinitely many of the a_n are positive integers (this is somewhat hand-wavy, but it makes sense if you think about the formal of definition of $\lim a_n = 0$).